Acta Numerica (1998), pp. 203–285

Stability for time-dependent differential equations

Heinz-Otto Kreiss*
Department of Mathematics,
UCLA, Los Angeles, CA 90095, USA
E-mail: kreiss@math.ucla.edu

Jens Lorenz†
Department of Mathematics and Statistics,
UNM, Albuquerque, NM 87131, USA
E-mail: lorenz@math.unm.edu

In this paper we review results on asymptotic stability of stationary states of PDEs. After scaling, our normal form is $u_t = Pu + \varepsilon f(u, u_x, \ldots) + F(x,t)$, where the (vector-valued) function $u(x,t)$ depends on the space variable x and time t. The differential operator P is linear, $F(x,t)$ is a smooth forcing, which decays to zero for $t \to \infty$, and $\varepsilon f(u, \ldots)$ is a nonlinear perturbation. We will discuss conditions that ensure $u \to 0$ for $t \to \infty$ when $|\varepsilon|$ is sufficiently small. If this holds, we call the problem asymptotically stable.

While there are many approaches to show asymptotic stability, we mainly concentrate on the resolvent technique. However, comparisons with the Lyapunov technique will also be given. The emphasis on the resolvent technique is motivated by the recent interest in pseudospectra.

* Supported by Office of Naval Research N 00014 92 J 1890.
† Supported by NSF Grant DMS-9404124 and DOE Grant DE-FG03-95ER25235.

CONTENTS

1. Introduction

A basic result in the stability theory of ODEs can be formulated as follows. If $y_0 \in \mathbb{R}^n$ is a fixed point of the ODE system

$$y_t \equiv \frac{\mathrm{d}y}{\mathrm{d}t} = \Phi(y), \tag{1.1}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field, then y_0 is asymptotically stable,[1] if all eigenvalues of the Jacobian $A = \Phi_y(y_0)$ have negative real parts. By definition, asymptotic stability of y_0 means the following.

(1) For all $\mu > 0$, there is $\varepsilon > 0$ so that[2] $|y_0 - y_1| < \varepsilon$ implies $|y_0 - y(t; y_1)| < \mu$ for all $t \geq 0$. Here $y(t; y_1)$ is the solution of (1.1) with $y = y_1$ at $t = 0$.
(2) There exists $\varepsilon_0 > 0$ such that $|y_0 - y(t; y_1)| \to 0$ as $t \to \infty$ whenever $|y_0 - y_1| < \varepsilon_0$.

Without going into details here (they are given in Section 2), the result can be made plausible. One introduces a new variable $u(t)$ by

$$y(t; y_1) = y_0 + \varepsilon u(t), \quad |y_0 - y_1| = \varepsilon,$$

[1] in the sense of Lyapunov (Lyapunov 1956)
[2] With $\langle u, v\rangle = \sum_j u_j v_j$ and $|u|^2 = \langle u, u\rangle$ we denote the Euclidean inner product and norm. The corresponding matrix norm is $|A| = \max\{|Au| : |u| = 1\}$.

for which one obtains

$$\begin{aligned}\varepsilon u_t &= \Phi(y_0 + \varepsilon u)\\ &= \varepsilon A u + \varepsilon^2 f(u), \quad |f(u)| \le C|u|^2.\end{aligned}$$

Therefore,

$$u_t = Au + \varepsilon f(u). \tag{1.2}$$

By assumption, all eigenvalues of A have negative real parts, which implies exponential decay of the solutions of the homogeneous equation $u_t = Au$. With proper arguments, the decay estimate can be extended to the nonlinear system (1.2) if $|\varepsilon|$ is sufficiently small, and asymptotic stability follows.

There are two kinds of difficulty in generalizing this basic stability result from ODEs to PDEs, one regarding the linear problem corresponding to $u_t = Au$, the other the small nonlinearity. To be more specific:

(1) In the PDE case, the linear operator corresponding to the matrix A might have a continuous spectrum and it might not be sufficient to look only at eigenvalues. Instead of exponential decay one might obtain only algebraic decay for solutions of the linear problem.
(2) In the PDE case, different norms enter the picture. For the linear problem, one might obtain decay of solutions in one norm, but not in another. Therefore, it is possible that a small nonlinear term $\varepsilon f(u)$ leads to asymptotic stability, whereas a term $\varepsilon f(u, u_x)$ does not.

Despite these difficulties, we will take the ODE case as a guideline. It will be convenient to generalize (1.2) to a system of the form

$$u_t = Au + \varepsilon f(t,u) + F(t) \tag{1.3}$$

where $f(t, 0) = 0$. For PDEs our corresponding normal form is

$$u_t = Pu + \varepsilon f(x, t, u, Du, \ldots, D^r u) + F(x, t). \tag{1.4}$$

Here x varies in a spatial domain Ω and $D^j u$ denotes the array of all spatial derivatives of $u = u(x, t)$ of order j.

The concept of asymptotic stability used in this paper is similar to Lyapunov's, but slightly more restrictive. For the ODE (1.3) our concept is as follows. We first consider the linear problem, obtained for $\varepsilon = 0$, with homogeneous initial condition

$$u = 0 \quad \text{at} \quad t = 0.$$

The forcing $F(t)$ will drive the system away from $u = 0$, and we ask if $u(t)$ will approach zero as $t \to \infty$ if $\lim_{t\to\infty} F(t) = 0$ (or if $F(t) \to 0$ as $t \to \infty$ at a certain rate). If this is so, we call (1.3) *linearly asymptotically stable.* If the same holds whenever $|\varepsilon|$ is sufficiently small, then we call (1.3) *nonlinearly asymptotically stable* (or simply *stable*).

For the ODE case, we will show in Section 2 that linear and also nonlinear asymptotic stability of (1.3) are both equivalent to the eigenvalue condition[3]

$$\operatorname{Re}\lambda < 0 \quad \text{for all} \quad \lambda \in \sigma(A). \tag{1.5}$$

In contrast, this eigenvalue condition is sufficient, but not necessary for asymptotic stability in the sense of Lyapunov of the zero solution of (1.3) with $F \equiv 0$. For example, for the equation $u_t = -u^3$ the zero solution is asymptotically stable in the sense of Lyapunov.

In short, the stability concept that we use here is slightly more restrictive than Lyapunov's, but also more robust. This makes it easier to generalize to PDEs, which is the main interest of the paper. In all cases, our sufficient conditions on asymptotic stability also provide estimates of the solution $u(x,t)$ for $0 \le t \le T$ by $F(x,t)$ for $0 \le t \le T$, in suitable norms, and the constant in the estimate will be independent of T and of $|\varepsilon| \le \varepsilon_0$. Therefore, asymptotic stability in Lyapunov's sense, where only the initial data are perturbed, can always be shown under our assumptions. See also Section 8.1.

(The introduction of the inhomogeneous term $F(t)$ can also be motivated by numerical considerations. When one interpolates the numerical values, one obtains a function u which satisfies a perturbed differential equation. The size of the perturbation is measured by $F(t)$.)

A more detailed outline of the paper follows. (Sections 2 to 5 are partially based on Kreiss and Lui (1997).) In Section 2 we use the Lyapunov technique and the resolvent technique (or Laplace-transform technique) to show asymptotic stability in the ODE setting if (1.5) holds. The solution-operator technique, which plays a major role for nonlinear wave equations, will be illustrated briefly in Section 2.3, but we refer to Racke (1992) for a comprehensive treatment.

In Sections 3 to 5 we assume that the linear operator P in (1.4) has constant coefficients and that the boundary conditions are periodic. Then, for the linear problem $u_t = Pu + F$, one can use Fourier expansions in space. The following two points will be emphasized.

(1) Assuming the Cauchy problem for $u_t = Pu + F$ to be well posed, the eigenvalues of the symbols $\hat{P}(\mathrm{i}\omega)$ tell what kind of resolvent estimate holds, *i.e.*, how many derivatives one gains when estimating u by F.

(2) If one gains q derivatives in the resolvent estimate, then the nonlinearity f in (1.4) may depend on all space derivatives of u of order $\le q$.

[3] With $\sigma(A)$ we denote the set of all eigenvalues of A.

For parabolic problems, these results are satisfactory. However, simple hyperbolic equations like

$$u_t = u_x - u + \varepsilon u u_x + F(x,t)$$

cannot be treated in this way, because one does not gain a derivative in the resolvent estimate. As we will show in Section 5 on hyperbolic problems, the Lyapunov technique can be applied to overcome the difficulty.

To give a problem where the resolvent technique is more adequate, consider the parabolic equation

$$u_t = Pu \equiv u_{xx} + (a(x)u)_x, \quad 0 \le x \le 1, \quad t \ge 0, \tag{1.6}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \ge 0, \tag{1.7}$$

and initial condition

$$u(x,0) = u_0(x), \quad 0 \le x \le 1.$$

The ODE operator

$$Pu = u_{xx} + (a(x)u)_x$$

has the adjoint $P^*u = u_{xx} - a(x)u_x$, which makes it easy to show by the maximum principle that all eigenvalues of P are negative. (Here we use the boundary conditions (1.7), for example.) Then, by our results in Section 6, the resolvent technique applies and asymptotic stability follows, even for fully nonlinear perturbations $\varepsilon f(x,t,u,u_x,u_{xx})$. If one wanted to obtain this result by the Lyapunov technique, one would have to construct an inner product $(\cdot,\cdot)_{\mathcal{H}}$ such that

$$(u, Pu)_{\mathcal{H}} \le -c(u,u)_{\mathcal{H}}, \quad c > 0.$$

However, it is not clear how to construct such an inner product, whose norm also has to be strong enough to bound u_{xx}.

It is natural to ask if one can combine the resolvent technique and the Lyapunov technique, using the strengths of each. This is indeed possible as demonstrated by Kreiss, Kreiss and Lorenz (1998*a*, 1998*b*). One area of applications is the study of mixed parabolic–hyperbolic systems. In the present paper we do not pursue this, however.

Sections 7 to 11 deal with PDEs on unbounded spatial domains Ω. In fact, we only consider the cases $\Omega = \mathbb{R}^d$ and $\Omega = \mathcal{H}^d$, a half-space. As will be shown in Sections 7 and 9, the unboundedness of the domain does not lead to difficulties as long as the linear operator P has a strictly negative zero-order term. A more interesting situation occurs when the zero-order term of P vanishes (or is semi-negative).

Using the resolvent technique, we will discuss this for parabolic problems on all space in Section 8 and will derive a weak resolvent estimate. The form of the linear estimate dictates more specific assumptions on the nonlinearity. The results apply to viscous conservation laws.

In Section 10 we consider the scalar parabolic model problem

$$u_t = \Delta u + a_1 u_{x_1} + \cdots + a_d u_{x_d} + G(x,t), \quad x_1 \geq 0, \tag{1.8}$$

on a half-space, again in the critical situation when the zero-order term vanishes, and show a weak resolvent estimate. The derivation is elementary but requires some rather technical analysis of ordinary BVPs on the half-line. It will become clear that the underlying hyperbolic part in (1.8) is important. In particular, the conditions for the weak resolvent estimate depend on the sign of the characteristic speed a_1 in relation to the boundary $x_1 = 0$.

Finally, in Section 11 we sketch stability results for parabolic systems on the real line, which are applicable to travelling waves.

The main emphasis in this paper is the derivation of stability results by the resolvent technique. Trefethen's work on pseudospectra (see Trefethen (1997) and the references given there) was a major motivation for this emphasis. Once a stability problem is cast into the normal form (1.4), one might want to answer the following questions.

(1) Applying Laplace transformation to the linear problem

$$u_t = Pu + F(x,t), \quad u(x,0) = 0, \tag{1.9}$$

one obtains the resolvent equation

$$(sI - P)\tilde{u} = \tilde{F}, \quad \operatorname{Re} s \geq 0. \tag{1.10}$$

What kind of estimates of $\tilde{u}$ by $\tilde{F}$ can one obtain? How do the estimates depend on s and how do they translate into estimates for physical variables?

(2) Given certain estimates for the linear problem (1.9), what kind of nonlinear problems (1.4) can one treat for small $|\varepsilon|$?

(3) How are the resolvent estimates for (1.10) related to properties of the spectrum or the pseudospectrum of P?

(4) If P depends on parameters – like the Reynolds number – how do the constants in the resolvent estimate scale as functions of the parameters and how does this affect the size $|\varepsilon|$ of the nonlinear terms for which one retains stability?

In this paper we only address the first two questions, though the other two are clearly of great interest. We remark that Romanov (1973) has considered plane-parallel Couette flow and has obtained an upper bound $-c\nu$ (with $c > 0$) for the real parts of the spectral values of a corresponding linearized operator P. Kreiss, Lundbladh and Henningson (1994*b*) have made attempts to obtain a bound for the resolvent $(sI - P)^{-1}$ and to address the fourth question for these flows. For the ODE case, we will sketch some simple related observations in Section 2.4.

2. Ordinary differential equations

Consider an initial value problem

$$u_t = Au + \varepsilon f(t,u) + F(t), \quad t \geq 0; \quad u(0) = u_0. \tag{2.1}$$

Here $A \in \mathbb{C}^{n\times n}$ is a constant matrix, the functions

$$f : \mathbb{C}^n \times [0,\infty) \to \mathbb{C}^n, \quad F : [0,\infty) \to \mathbb{C}^n$$

are assumed to be C^∞, for simplicity, and $u_0 \in \mathbb{C}^n$ is a given initial vector. We will always assume that $F(t)$ is a bounded function and set

$$|F|_\infty = \sup_{t\geq 0} |F(t)|.$$

The function $f(t,u)$ is assumed to vanish at $u = 0$. More precisely, we assume that

$$\text{for all } c_1 > 0 \text{ there exists } C_1 > 0 \text{ with } \quad |f(u,t)| \leq C_1|u| \quad \text{if } |u| \leq c_1. \tag{2.2}$$

Our concept of asymptotic stability is as follows.

Definition 2.1 Problem (2.1) with $\varepsilon = 0$ is called linearly asymptotically stable if

$$\lim_{t\to\infty} F(t) = 0 \quad \text{implies} \quad \lim_{t\to\infty} u(t) = 0. \tag{2.3}$$

Furthermore, (2.1) is called nonlinearly asymptotically stable if (2.3) holds for all sufficiently small $|\varepsilon|$.

If A has an eigenvalue λ with $\operatorname{Re}\lambda \geq 0$ and $A\phi = \lambda\phi, \phi \neq 0$, then $u(t) = \mathrm{e}^{\lambda t}\phi$ solves $u_t = Au$, but $u(t)$ does not tend to zero as $t \to \infty$. Therefore, the following condition is necessary for asymptotic stability of (2.1).

Eigenvalue condition.

$$\operatorname{Re}\lambda < 0 \quad \text{for all} \quad \lambda \in \sigma(A). \tag{2.4}$$

We will now show that (2.4) *characterizes* asymptotic stability.

Theorem 2.1 Under assumption (2.2) the eigenvalue condition (2.4) is necessary and sufficient for linear (and also for nonlinear) asymptotic stability of (2.1).

For illustration, we will prove Theorem 2.1 in two different ways, namely the Lyapunov technique (or energy estimate) and the resolvent technique.[4]

[4] The assumption for $F(t)$ is slightly different in the result proved by the resolvent technique: see Theorem 2.3.

In the ODE case, both techniques lead to the same characterization of asymptotically stable problems (Theorem 2.1). In the PDE case, the two techniques have different strengths and weaknesses, however, as we will show in the subsequent sections.

2.1. The Lyapunov technique

The linear problem

Let us first show linear asymptotic stability under assumption (2.4). The easiest case occurs if A is normal, where we can use the following result.[5]

Lemma 2.1 Let A be normal and let

$$\operatorname{Re}\lambda \le -\delta < 0 \quad \text{for all} \quad \lambda \in \sigma(A). \tag{2.5}$$

Then we have

$$A + A^* \le -2\delta I. \tag{2.6}$$

Proof. There is a unitary matrix U such that

$$U^*AU = \Lambda = \operatorname{diag}(\lambda_j), \quad \operatorname{Re}\lambda_j \le -\delta.$$

Therefore,

$$A + A^* = U(\Lambda + \bar{\Lambda})U^* \le -2\delta I.$$

□

Now assume that A is normal and let $u_t = Au + F(t)$. Then we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 &= \langle u, u_t\rangle + \langle u_t, u\rangle \\
&= \langle u, (A + A^*)u\rangle + \langle u, F\rangle + \langle F, u\rangle \\
&\le -2\delta|u|^2 + 2|u||F| \\
&\le -\delta|u|^2 + \frac{1}{\delta}|F|^2.
\end{aligned}$$

This implies

$$\begin{aligned}
|u(t)|^2 &\le \mathrm{e}^{-\delta t}|u_0|^2 + \frac{1}{\delta}\int_0^t \mathrm{e}^{-\delta(t-\tau)}|F(\tau)|^2\,\mathrm{d}\tau \\
&\le \mathrm{e}^{-\delta t}|u_0|^2 + \frac{1}{\delta^2}\max_{0\le\tau\le t}|F(\tau)|^2;
\end{aligned} \tag{2.7}$$

thus

$$|u(t)|^2 \le |u_0|^2 + \frac{1}{\delta^2}|F|_\infty^2 =: c_0^2, \quad t \ge 0. \tag{2.8}$$

[5] If $H_1, H_2 \in \mathbb{C}^{n\times n}$ are Hermitian matrices, then we write $H_1 \le H_2$ if and only if $u^*H_1u \le u^*H_2u$ for all $u \in \mathbb{C}^n$. Similarly, we write $H_1 < H_2$ if and only if $u^*H_1u < u^*H_2u$ for all $u \in \mathbb{C}^n, u \ne 0$.

Clearly, we may start the estimate at any t_0 and obtain

$$|u(t)|^2 \leq \mathrm{e}^{-\delta(t-t_0)}|u(t_0)|^2 + \frac{1}{\delta^2} \max_{t_0 \leq \tau \leq t} |F(\tau)|^2. \tag{2.9}$$

Now recall the assumption $\lim_{t\to\infty} F(t) = 0$. If $\eta > 0$ is given, there exists $t_0(\eta)$ with

$$\frac{1}{\delta^2} \sup_{t_0(\eta) \leq \tau < \infty} |F(\tau)|^2 \leq \eta^2.$$

Therefore, (2.9) and (2.8) imply

$$|u(t)|^2 \leq 2\eta^2 \quad \text{for} \quad t \geq t_1(\eta),$$

and we have shown $\lim_{t\to\infty} u(t) = 0$.

If A is not normal, the crucial inequality $A + A^* \leq -2\delta I$ does not follow from (2.5), in general. However, by changing the inner product, we can basically still argue as before. The following notation will be used.

Notation

If $H > 0$ is a positive definite Hermitian matrix, then a scalar product and norm are determined by

$$\langle u, v\rangle_H = u^* H v, \quad |u|_H^2 = u^* H u, \quad u, v \in \mathbb{C}^n. \tag{2.10}$$

The corresponding matrix norm is

$$|B|_H = \max_{|u|_H = 1} |Bu|_H.$$

We note that the inequalities

$$\frac{1}{c} I \leq H \leq cI, \quad c \geq 1, \tag{2.11}$$

are equivalent to the norm estimates

$$\frac{1}{c}\, |u|^2 \leq |u|_H^2 \leq c|u|^2 \quad \text{for all} \quad u \in \mathbb{C}^n, \tag{2.12}$$

and therefore (2.11) implies

$$\frac{1}{c}\, |B|_H \leq |B| \leq c|B|_H, \quad B \in \mathbb{C}^{n\times n}. \tag{2.13}$$

Lemma 2.1 has the following generalization to arbitrary matrices $A \in \mathbb{C}^{n\times n}$.

Theorem 2.2 If

$$\operatorname{Re} \lambda \leq -\delta < -\delta_1 < 0 \quad \text{for all} \quad \lambda \in \sigma(A), \tag{2.14}$$

then there exists a positive definite Hermitian matrix H with

$$HA + A^* H \leq -2\delta_1 H < 0. \tag{2.15}$$

Proof. By Schur's theorem there exists a unitary matrix U so that

$$U^*AU = \Lambda + R = \operatorname{diag}(\lambda_j) + R, \quad \operatorname{Re}\lambda_j \le -\delta < 0,$$

where R is strictly upper triangular. We set

$$D = \operatorname{diag}(1, \varepsilon, \dots, \varepsilon^{n-1}), \quad \varepsilon > 0,$$

and consider

$$D^{-1}U^*AUD = \Lambda + D^{-1}RD.$$

The entries in $D^{-1}RD$ are $\mathcal{O}(\varepsilon)$. Setting $S = (UD)^{-1}$ one obtains

$$\begin{aligned} SAS^{-1} + (SAS^{-1})^* &= \Lambda + \bar{\Lambda} + \mathcal{O}(\varepsilon) \\ &\le -2\delta_1 I \end{aligned}$$

for sufficiently small ε. Therefore, if one defines $H = S^*S$, one obtains the desired inequality

$$\begin{aligned} HA + A^*H &= S^*(\Lambda + \bar{\Lambda} + \mathcal{O}(\varepsilon))S \\ &\le -2\delta_1 H \end{aligned}$$

for sufficiently small $\varepsilon > 0$. □

Using $|\cdot|_H$ one can estimate the solutions $u(t)$ of $u_t = Au + F(t)$ as before,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|u|_H^2 &= \langle u, u_t\rangle_H + \langle u_t, u\rangle_H \\ &= \langle u, (HA + A^*H)u\rangle + \langle u, F\rangle_H + \langle F, u\rangle_H \\ &\le -2\delta_1 |u|_H^2 + 2|u|_H |F|_H \\ &\le -\delta_1 |u|_H^2 + \frac{1}{\delta_1}|F|_H^2. \end{aligned}$$

Proceeding in the same way as above, we find that $u(t) \to 0$ as $t \to \infty$. Here the equivalence of the norms $|\cdot|$ and $|\cdot|_H$ is used. The equivalence is valid, of course, since we work in a finite dimensional setting. Let us note, however, that the inclusion (2.11) leads to an equivalence constant; see (2.12). This observation will be useful for PDEs, where we deal with infinite families of matrices.

The nonlinear problem

Now consider the nonlinear problem (2.1). For simplicity, let A be normal so that we can use the Euclidean norm for our estimates. In the general case, the same arguments apply with $|\cdot|_H$.

Recall the estimate (2.8) for $\varepsilon = 0$, where $c_0 > 0$ without loss of generality. By continuity, for any ε there exists $T_\varepsilon > 0$ such that

$$|u(t)|^2 \le 3c_0^2 =: c_1^2 \quad \text{for} \ \ 0 \le t \le T_\varepsilon. \tag{2.16}$$

Using (2.2) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 &\le -2\delta|u|^2 + 2|\varepsilon||u||f(t,u)| + 2|u||F| \\ &\le -\delta|u|^2 + \frac{1}{\delta}|F|^2 + 2|\varepsilon|C_1|u|^2 \qquad (2.17)\end{aligned}$$

in $0 \le t \le T_\varepsilon$. Henceforth, assume that $|\varepsilon|$ is so small that

$$2|\varepsilon|C_1 \le \frac{\delta}{2}. \qquad (2.18)$$

Then we find

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 \le -\frac{\delta}{2}|u|^2 + \frac{1}{\delta}|F|^2;$$

thus

$$\begin{aligned}|u(t)|^2 &\le \mathrm{e}^{-\delta t/2}|u_0|^2 + \frac{1}{\delta}\int_0^t \mathrm{e}^{-\delta(t-\tau)/2}|F(\tau)|^2\,\mathrm{d}\tau \\ &\le |u_0|^2 + \frac{2}{\delta^2}\max_{0\le\tau\le t}|F(\tau)|^2 \\ &\le 2c_0^2. \qquad (2.19)\end{aligned}$$

Therefore, always assuming (2.18), we have shown that $|u(t)|^2 \le 3c_0^2$ in $0 \le t \le T_\varepsilon$ implies $|u(t)|^2 \le 2c_0^2$ in $0 \le t \le T_\varepsilon$. A simple continuation and contradiction argument yields that $u(t)$ exists for all $t \ge 0$ and

$$|u(t)| \le 2c_0^2 \quad \text{for all} \quad t \ge 0.$$

With the same arguments as before, we find

$$|u(t)|^2 \le \mathrm{e}^{-\delta(t-t_0)/2}2c_0^2 + \frac{2}{\delta^2}\sup_{t_0\le\tau<\infty}|F(\tau)|^2,$$

and therefore (2.3) holds. This proves Theorem 2.1.

2.2. The resolvent technique

Consider again the ODE (2.1). To illustrate the resolvent technique, we will prove the following result.

Theorem 2.3 Assume (2.2), (2.4) and let $F \in L_2$. Then $\lim_{t\to\infty} u(t) = 0$ if $|\varepsilon|$ is sufficiently small.

It will be convenient to have homogeneous initial data, which can always be achieved by applying the simple transformation[6]

$$u(t) = \mathrm{e}^{-t}u_0 + v(t).$$

[6] For generalizations, it is important to note that we do not make use of the exponential decay of e^{At}.

The function $v(t)$ satisfies $v(0) = 0$ and

$$v_t = Av + \varepsilon\Big(f(t, \mathrm{e}^{-t}u_0 + v) - f(t, \mathrm{e}^{-t}u_0)\Big) + G(t)$$

with

$$G(t) = F(t) + \mathrm{e}^{-t}(Au_0 + u_0) + \varepsilon f(t, \mathrm{e}^{-t}u_0), \quad \lim_{t\to\infty} G(t) = 0.$$

If we set

$$\phi(\xi) = f(t, \mathrm{e}^{-t}u_0 + \xi v), \quad 0 \le \xi \le 1,$$

we can write

$$\begin{aligned} f(t, \mathrm{e}^{-t}u_0 + v) - f(t, \mathrm{e}^{-t}u_0) &= \phi(1) - \phi(0) = \int_0^1 \phi'(\xi)\,\mathrm{d}\xi \\ &= \left(\int_0^1 f_u(t, \mathrm{e}^{-t}u_0 + \xi v)\,\mathrm{d}\xi\right) v \\ &=: g(t, v). \end{aligned} \tag{2.20}$$

Thus, for v we obtain a transformed equation

$$v_t = Av + \varepsilon g(t, v) + G(t), \quad v(0) = 0,$$

of similar form with homogeneous initial data.

For notational convenience, we will assume that the initial data in (2.1) are already homogeneous.

Properties of the Laplace transform

Let us first recall some elementary properties of the Laplace transform. If $g(t)$ is a continuous function of $t \ge 0$ with values in $\mathbb{C}^n$, which satisfies a growth restriction

$$|g(t)| \le K\mathrm{e}^{\alpha t}, \quad t \ge 0,$$

then its Laplace transform is the analytic function

$$\tilde{g}(s) = \int_0^\infty \mathrm{e}^{-st} g(t)\,\mathrm{d}t, \quad \operatorname{Re} s > \alpha.$$

The inverse transform reads

$$g_0(t) = \frac{1}{2\pi i}\int_{\eta - \mathrm{i}\infty}^{\eta + \mathrm{i}\infty} \mathrm{e}^{st}\tilde{g}(s)\,\mathrm{d}s, \quad \eta > \alpha,$$

where

$$g_0(t) = g(t) \quad \text{for} \quad t > 0, \quad g_0(0) = \frac{1}{2}g(0), \quad g_0(t) = 0 \quad \text{for} \quad t < 0.$$

Here the path of integration is $\Gamma_\eta = \{s = \eta + \mathrm{i}\xi, -\infty < \xi < \infty\}$.

Parseval's relation reads

$$\int_0^\infty \mathrm{e}^{-2\eta t}|g(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^\infty |\tilde{g}(\eta+\mathrm{i}\xi)|^2\,\mathrm{d}\xi, \quad \eta > \alpha. \tag{2.21}$$

If $g \in C^1$ and

$$|g(t)| + |g_t(t)| \leq K\mathrm{e}^{\alpha t}, \quad t \geq 0,$$

then

$$\tilde{g}_t(s) = s\tilde{g}(s) - g(0), \quad \operatorname{Re} s > \alpha,$$

as follows directly from the definition through integration by parts.

Estimates in the linear case

Now consider (2.1) with $\varepsilon = 0, u_0 = 0$, and first assume that $F(t) = 0$ for $t > T$. Then $\tilde{F}(s)$ and $\tilde{u}(s)$ are well defined for $\operatorname{Re} s \geq 0$. Laplace transformation gives us

$$s\tilde{u}(s) = A\tilde{u}(s) + \tilde{F}(s), \quad \operatorname{Re} s \geq 0; \tag{2.22}$$

thus

$$\tilde{u}(s) = (sI - A)^{-1}\tilde{F}(s), \quad \operatorname{Re} s \geq 0. \tag{2.23}$$

For any $A \in \mathbb{C}^{n\times n}$ the matrix-valued function

$$(sI - A)^{-1}, \quad s \in \mathbb{C} \setminus \sigma(A),$$

is called the resolvent of A. We list some of its elementary properties.

Lemma 2.2

(a) For any $A \in \mathbb{C}^{n\times n}$ we have

$$|(sI - A)^{-1}| \leq \frac{1}{|s| - |A|} \quad \text{if } |s| > |A|.$$

(b) If $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$, then

$$R := \sup_{\operatorname{Re} s \geq 0} |(sI - A)^{-1}| \tag{2.24}$$

is finite. By definition, R is the resolvent constant of A.

(c) If A is normal and

$$\max_{\lambda\in\sigma(A)} \operatorname{Re}\lambda =: -\delta < 0,$$

then $R = \frac{1}{\delta}$.

Proof.

(a) Let $|s| > |A|$ and let

$$sv = Av + b, \quad b, v \in \mathbb{C}^n.$$

Then one obtains $|s||v| \leq |A||v| + |b|$; thus $|v| \leq (|s| - |A|)^{-1}|b|$.

(b) Define $\Omega = \{s \in \mathbb{C} \ : \ \operatorname{Re} s \geq 0, \ |s| \leq |A| + 1\}$. By compactness,

$$\max_{s \in \Omega} |(sI - A)^{-1}| =: R_1$$

is finite. Together with (a) one obtains $R \leq \max\{R_1, 1\}$.

(c) There is a unitary matrix U such that $U^* A U = \Lambda = \operatorname{diag}(\lambda_j)$ is diagonal. If

$$s = \eta + \mathrm{i}\xi, \quad \eta \geq 0, \quad \lambda_j = \alpha_j + \mathrm{i}\beta_j, \quad \alpha_j \leq -\delta,$$

then one obtains

$$\begin{aligned} |(sI - A)^{-1}|^2 &= |(sI - \Lambda)^{-1}|^2 \\ &= \max_j \frac{1}{|s - \lambda_j|^2} \\ &= \max_j \frac{1}{(\eta - \alpha_j)^2 + (\xi - \beta_j)^2} \\ &\leq \frac{1}{\delta^2}. \end{aligned}$$

In the last estimate equality holds for $s = \mathrm{i}\beta_j$ if $\operatorname{Re} \alpha_j = -\delta$. This proves the lemma. □

Assuming the eigenvalue condition (2.4), we obtain from (2.23)

$$|\tilde{u}(s)| \leq R \, |\tilde{F}(s)|, \quad \operatorname{Re} s \geq 0.$$

Then Parseval's relation (see (2.21) and set $\eta = 0$) implies that

$$\int_0^\infty |u(t)|^2 \, \mathrm{d}t \leq R^2 \int_0^\infty |F(t)|^2 \, \mathrm{d}t. \tag{2.25}$$

So far, we have assumed that $F(t)$ vanishes for large t. However, if $F \in L_2$ is arbitrary, we can approximate F by a sequence F_n with $F_n(t) = F(t)$ for $t \leq n$ and $F_n(t) = 0$ for $t \geq n + 1$. A simple limit argument shows that the estimate (2.25) still holds.

Remark 2.1 Using the definition of the resolvent constant R and Parseval's relation, it is not difficult to show that R is the *best* constant for which (2.25) holds for all $F \in L_2$.

The estimate (2.25) is not strong enough to yield a nonlinear stability result, because one cannot bound point values $u(t)$ in terms of the L_2-integral.

An estimate of point values is generally necessary, however, to control nonlinearities $f(t,u)$. We can strengthen the estimate (2.25) as follows.

Theorem 2.4 Consider

$$u_t = Au + F(t), \quad t \geq 0, \quad u(0) = 0,$$

and assume the eigenvalue condition (2.4). There is a constant K_1, depending only on A, so that

$$\int_0^\infty \left(|u(t)|^2 + |u_t(t)|^2\right) \mathrm{d}t \leq K_1 \int_0^\infty |F(t)|^2 \,\mathrm{d}t. \tag{2.26}$$

Proof. As before, we first assume $F(t) = 0$ for large t. Since $u(0) = 0$, Laplace transformation yields

$$\tilde{u}_t(s) = s\tilde{u}(s) = A\tilde{u}(s) + \tilde{F}(s);$$

thus

$$\begin{aligned} |\tilde{u}_t(s)| &\leq |A||\tilde{u}(s)| + |\tilde{F}(s)| \\ &\leq (|A|R+1)|\tilde{F}(s)|. \end{aligned}$$

Using the same arguments as above, the desired estimate follows. □

Since values of $F(t)$ for $t > T$ do not affect the solution $u(t)$ for $t \leq T$, it is not difficult to show that the estimate (2.26) can be restricted to any finite time interval, that is,

$$\int_0^T \left(|u(t)|^2 + |u_t(t)|^2\right) \mathrm{d}t \leq K_1 \int_0^T |F(t)|^2 \,\mathrm{d}t. \tag{2.27}$$

The left-hand side of (2.27) can now be used to bound u in maximum norm. This follows easily from $u(0) = 0$ and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|u|^2 &= \langle u, u_t\rangle + \langle u_t, u\rangle \\ &\leq 2|u||u_t| \leq |u|^2 + |u_t|^2. \end{aligned}$$

The resulting estimate is a simple example of a Sobolev inequality, which we state next.

Lemma 2.3 Let $u(t), a \leq t \leq b$, denote a C^1-function. Then we have

$$\max_{a\leq t\leq b} |u(t)|^2 \leq \left(1 + \frac{1}{b-a}\right) \int_a^b |u|^2 \,\mathrm{d}t \; + \int_a^b |u_t|^2 \,\mathrm{d}t. \tag{2.28}$$

If $u(t^*) = 0$ for some $a \leq t^* \leq b$, then

$$\max_{a\leq t\leq b} |u(t)|^2 \leq \int_a^b \left(|u|^2 + |u_t|^2\right) \mathrm{d}t. \tag{2.29}$$

Applying (2.28), for $b \to \infty$, one obtains that

$$\sup_{t\geq t_0} |u(t)|^2 \leq \int_{t_0}^{\infty} \left(|u|^2 + |u_t|^2\right) \mathrm{d}t. \tag{2.30}$$

The right-hand side tends to zero for $t_0 \to \infty$ since $\int_0^\infty (|u|^2 + |u_t|^2)\,\mathrm{d}t$ is finite by Theorem 2.4. Thus we have shown the claim of Theorem 2.3 for $\varepsilon = 0$.

Nonlinear stability

Since our estimates are strong enough to control u in maximum norm, it is not difficult to extend the result to small $|\varepsilon|$. The arguments are as follows.

For any ε there exists $T_\varepsilon > 0$ with

$$\int_0^{T_\varepsilon} \left(|u|^2 + |u_t|^2\right) \mathrm{d}t \leq 4K_1 \int_0^\infty |F|^2 \,\mathrm{d}t =: c_1^2. \tag{2.31}$$

By (2.29),

$$|u(t)| \leq c_1, \quad 0 \leq t \leq T_\varepsilon,$$

thus

$$|f(t, u(t))| \leq C_1 |u(t)|, \quad 0 \leq t \leq T_\varepsilon,$$

and therefore

$$\int_0^{T_\varepsilon} |f(t,u(t))|^2 \,\mathrm{d}t \leq C_1^2 \int_0^{T_\varepsilon} |u|^2 \,\mathrm{d}t. \tag{2.32}$$

Application of (2.27), with $F(t)$ replaced by $F(t) + \varepsilon f(t, u(t))$, yields

$$\int_0^{T_\varepsilon} \left(|u|^2 + |u_t|^2\right) \mathrm{d}t \leq 2K_1 \int_0^{T_\varepsilon} \left(|F|^2 \;+\; |\varepsilon|^2 C_1^2 |u|^2\right) \,\mathrm{d}t.$$

Assuming that $|\varepsilon|$ is so small that

$$2K_1 |\varepsilon|^2 C_1^2 \leq \frac{1}{3},$$

we obtain

$$\int_0^{T_\varepsilon} \left(|u|^2 + |u_t|^2\right) \mathrm{d}t \leq 3K_1 \int_0^\infty |F|^2 \,\mathrm{d}t = \frac{3}{4} c_1^2. \tag{2.33}$$

To summarize, from (2.31) we could conclude (2.33) under the above smallness assumption for $|\varepsilon|$. A simple continuation and contradiction argument yields existence of $u(t)$ for all $t \geq 0$, and

$$\int_0^\infty \left(|u|^2 + |u_t|^2\right) \mathrm{d}t \leq \frac{3}{4} c_1^2 < \infty.$$

As before, $\lim_{t\to\infty} u(t) = 0$ follows.

Outline of generalizations to PDEs
It is worthwhile to re-emphasize the main points of the above arguments and to indicate generalizations. There are essentially three steps.

(1) An estimate for a linear problem $u_t = Pu + F(t)$ of the type

$$\|u\|_U \le K_1 \|F\|_V. \tag{2.34}$$

Such an estimate will generalize (2.26).

(2) A Sobolev inequality like

$$|u|_\infty \le K_2 \|u\|_U. \tag{2.35}$$

Such an estimate will generalize (2.29). In the PDE case, if the nonlinearity f depends also on u_x, *etc.*, then $\|u\|_U$ has to be strong enough to bound $|u_x|_\infty$, *etc.*, too.

(3) An estimate of the nonlinear term of the following type. If $|u|_\infty \le \kappa$ then

$$\|f(t, u(t))\|_V \le C_\kappa \|u\|_U. \tag{2.36}$$

Such an estimate will generalize (2.32).

Now assume we have established (2.34), (2.35), (2.36). Then, in order to bound the solution of the nonlinear problem

$$u_t = Pu + \varepsilon f(t, u) + F(t)$$

for small $|\varepsilon|$, we formally proceed as follows. We consider $\varepsilon f(t, u(t))$ as part of the forcing so that (2.34) yields

$$\|u\|_U \le K_1(\|F\|_V + |\varepsilon|\ \|f\|_V). \tag{2.37}$$

Assuming, tentatively, that

$$\|u\|_U \le 2K_1 \|F\|_V, \tag{2.38}$$

one obtains from (2.35)

$$|u|_\infty \le 2K_1 K_2 \|F\|_V =: \kappa.$$

Then (2.36) implies

$$\|f(t, u(t))\|_V \le C_\kappa \|u\|_U$$

and (2.37) yields

$$\|u\|_U \le K_1(\|F\|_V + |\varepsilon|\ C_\kappa\ \|u\|_U).$$

If ε satisfies the restriction

$$|\varepsilon| K_1 C_\kappa \le \frac{1}{3},$$

then the previous estimate gives us the desired bound

$$\|u\|_U \le \frac{3}{2} K_1 \|F\|_V,$$

which is consistent with the tentative assumption (2.38). As in the proof of Theorem 2.4, these formal arguments can be made rigorous by restricting the relevant estimates to finite intervals $0 \le t \le T$. Here it is assumed that the norm $\|u\|_U$ is strong enough to guarantee continuation of a local solution.

2.3. The solution-operator technique

There are cases where it is best to argue directly with the solution operator of the linear homogeneous equation. For illustration, we consider the ODE initial value problem

$$u_t = -\frac{\gamma}{t+1}\, u + \varepsilon u^\rho + F(t), \quad u(0) = u_0, \tag{2.39}$$

with constants $\gamma > 0, \rho \ge 1$, and a continuous forcing $F(t)$ which satisfies an estimate

$$|F(t)| \le \frac{K}{(t+1)^\beta}, \quad \beta > 0. \tag{2.40}$$

We ask for conditions on β and ρ which imply that, for sufficiently small $|\varepsilon|$, $u(t)$ converges to zero for $t \to \infty$ as fast as the solution of the homogeneous equation

$$u_t = -\frac{\gamma}{t+1}\, u, \quad u(0) = u_0. \tag{2.41}$$

The solution of (2.41) is

$$u_h(t) = (t+1)^{-\gamma} u_0,$$

and the solution of (2.39) with $\varepsilon = 0$ is

$$\begin{aligned} u(t) &= (t+1)^{-\gamma} u_0 + \int_0^t \left(\frac{\xi+1}{t+1}\right)^\gamma F(\xi)\, \mathrm{d}\xi \\ &= : u_h(t) + J(t). \end{aligned}$$

Using (2.40) we can bound the integral term as follows:

$$|J(t)| \le K(t+1)^{-\gamma} \int_0^t (\xi+1)^{\gamma-\beta}\, \mathrm{d}\xi,$$

which shows that the solution $u(t)$ decays like $(t+1)^{-\gamma}$ if $\gamma - \beta < -1$.

Now consider the nonlinear problem (2.39). Writing εu^ρ as forcing, one obtains

$$\begin{aligned}\phi(t) := (t+1)^\gamma |u(t)| &\leq |u_0| + K' + |\varepsilon| \int_0^t (\xi+1)^\gamma |u(\xi)|^\rho \,\mathrm{d}\xi \\ &\leq |u_0| + K' + |\varepsilon| \left(\max_{0\leq\xi\leq t} \phi(x) \right) \int_0^t (\xi+1)^{\gamma-\gamma\rho} \,\mathrm{d}\xi.\end{aligned}$$

The integral is finite if $\gamma - \gamma\rho < -1$. Thus, if one makes the assumptions

$$\beta > 1+\gamma \quad \text{and} \quad \rho > 1 + \frac{1}{\gamma},$$

then the function $\phi(t)$ remains bounded if $|\varepsilon|$ is sufficiently small and, therefore, $|u(t)| \leq C(t+1)^{-\gamma}$.

For certain classes of PDEs the solution operator technique is very powerful. This is true, in particular, if an explicit solution for the linear part of the equation is available, which yields accurate estimates of the solution operator. We refer to the book by Racke (1992) for applications to nonlinear wave equations and other systems of PDEs on all space. Earlier references with similar ideas are Kawashima (1987), Klainerman and Ponce (1983), Matsumura and Nishida (1979), Shatah (1982), and Strauss (1981).

2.4. Remarks on the size of perturbations

Remark 2.2 An important and interesting problem is to quantify the size of the perturbation that one is allowed to apply to a stable system without losing stability. This has been emphasized in the recent work of Trefethen (1997). In general, the question is difficult, and any answer will depend on the norms that are used. Some insight can be obtained as follows. Assuming that $u_t = Au + F(t)$ is linearly stable, we add a perturbation $\varepsilon f(u) = \varepsilon Bu$, which is also linear. (See Remark 2.3 below for nonlinear perturbations.) Laplace transformation yields

$$s\tilde{u} = A\tilde{u} + \varepsilon B\tilde{u} + \tilde{F}; \tag{2.42}$$

thus

$$\tilde{u} = \varepsilon(sI-A)^{-1}B\tilde{u} + (sI-A)^{-1}\tilde{F}.$$

Recall that $R = \sup_{\mathrm{Re}\, s\geq 0} |(sI-A)^{-1}|$ is the resolvent constant of A. If

$$|\varepsilon B| R \leq \frac{1}{2}, \tag{2.43}$$

say, then $|\tilde{u}| \leq 2R\|\tilde{F}|$, and we obtain essentially the same estimate of u by F as in the unperturbed case. Thus, if (2.43) holds, stability is retained. On the other hand, if $|\varepsilon B|R \geq 1$, then the system (2.42) may be singular, allowing for instability, and we conclude that (2.43) is a realistic condition for retaining stability. This shows the importance of the resolvent constant.

How is the resolvent constant R related to the eigenvalues of A? Trefethen defines the spectral abscissa of A by

$$\alpha(A) = \max_{\lambda \in \sigma(A)} \operatorname{Re} \lambda.$$

Then, if A is normal and $\alpha(A) = -\delta < 0$, we have $R = \frac{1}{\delta}$, as noted in Lemma 2.2(c). In this case, condition (2.43) reads

$$|\varepsilon B| \le \frac{1}{2}\delta. \tag{2.44}$$

If A is not normal, then $R \gg \frac{1}{\delta}$ is possible, as emphasized by Trefethen (1997). Clearly, if $R \gg \frac{1}{\delta}$, the requirement (2.43) is much more restrictive than (2.44). For this reason it is important to obtain good estimates for the resolvent constant R. The techniques for proving the Kreiss matrix theorem (Kreiss and Lorenz 1989) can be applied, in principle, but the treatment of concrete examples may be formidable. For the alternative approach of computing the pseudospectrum of A numerically, we refer to Trefethen (1997). (If $\sigma_\varepsilon(A) = \{z \in \mathbb{C} : |(zI - A)^{-1}| \ge 1/\varepsilon\}$ denotes the ε-pseudospectrum of A, then $R = 1/\varepsilon_0$, where ε_0 $\{\varepsilon > 0 : \sigma_\varepsilon(A)$ lies in the left half-plane$\}$.)

Remark 2.3 Consider the linear problem $u_t = Au + F(t), u(0) = 0$. If $F(0) = 0$ (which can always be enforced by the transformation $u = te^{-t}F(0) + v$), we obtain from (2.25) and $u_{tt} = Au_t + F_t$ the estimate

$$\int_0^\infty \left(|u|^2 + |u_t|^2\right) \mathrm{d}t \le R^2 \int_0^\infty \left(|F|^2 + |F_t|^2\right) \mathrm{d}t. \tag{2.45}$$

The left-hand side bounds $\sup_t |u(t)|^2$, and we can use (2.45) instead of (2.26) to show nonlinear stability. Then our arguments, given above, show that the size of R is again crucial for determining the size of $|\varepsilon|$ which retains stability for the nonlinear perturbed problem.

3. Parabolic systems: periodic boundary conditions

Consider a parabolic equation

$$u_t = Pu + \varepsilon f\left(x, t, u, Du, D^2u\right) + F(x,t), \quad x \in \mathbb{R}^d, \quad t \ge 0, \tag{3.1}$$

where P is a constant coefficient operator,

$$Pu = \Delta u + \sum_{j=1}^{d} A_j D_j u + Bu. \tag{3.2}$$

Here

$$D_j = \frac{\partial}{\partial x_j}, \quad \Delta = D_1^2 + \cdots + D_d^2, \tag{3.3}$$

and $A_j, B \in \mathbb{C}^{n\times n}$ are constant matrices. The function $u(x,t)$ takes values in $\mathbb{C}^n$. For simplicity, f and F are assumed to be C^∞ functions of their arguments; the nonlinearity f may depend on x, t, u and

$$Du = (D_1 u, \ldots, D_d u), \qquad D^2 u = (D_i D_j u)_{1\le i,j\le d} \; .$$

A main assumption is that $F(x,t)$ and $f(x,t,u,Du,D^2u)$ are 2π-periodic in each x_j, and we seek a solution $u(x,t)$ with the same spatial periodicity property. In other words, the space variable x lives in the d-torus $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$.

We want to discuss asymptotic stability using the Laplace transform (or resolvent) technique and will assume, without loss of generality, a homogeneous initial condition

$$u(x,0) = 0, \quad x \in \mathbb{R}^d. \tag{3.4}$$

Setting

$$v = (u, Du, D^2u),$$

we will assume that $f(x,t,v)$ vanishes at $v = 0$. More precisely, with an integer p specified below, we require the following.

Assumption 3.1 For all $c_1 > 0$ there exists $C_1 > 0$ with

$$\begin{aligned} |D_x^\beta f(x,t,v)| &\le C_1|v| \quad \text{if } |v| \le c_1, \ |\beta| \le p, \\ |D_x^\beta D_v^\gamma f(x,t,v)| &\le C_1 \quad \text{if } |v| \le c_1, \ |\beta| + |\gamma| \le p, \ |\gamma| \ge 1. \end{aligned} \tag{3.5}$$

We will show nonlinear asymptotic stability of (3.1) in the sense of the following theorem.[7]

Theorem 3.1 Let Assumption 3.1 hold with $p = d + 5$, and let

$$\int_0^\infty \|F(\cdot,t)\|_{H^p}^2 \, dt < \infty \tag{3.6}$$

be finite. If $|\varepsilon|$ is small enough, then $\max_x |u(x,t)| \to 0$ as $t \to \infty$.

Remark 3.1 The value $p = d + 5$ is not optimal; that is, it suffices to require Assumption 3.1 and (3.6) with a smaller value of p. For most applications this is uninteresting, since the assumptions hold for all p if they hold for some small p. See also Remark 3.2 on page 228.

[7] We use the standard notation $\|u\|_{H^p}^2 = \sum_{|\alpha|\le p} \|D^\alpha u\|^2$ where $\|u\|^2 = \int_{\mathbb{T}^d} |u(x)|^2 \, dx$, $D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}$ and $|\alpha| = \sum_j \alpha_j$.

The linear problem
To begin with, consider the linear problem $u_t = Pu + F$. Applying Fourier expansion in space, we obtain

$$\hat{u}_t(\omega, t) = \hat{P}(\mathrm{i}\omega)\hat{u}(\omega, t) + \hat{F}(\omega, t), \quad \omega \in \mathbb{Z}^d. \tag{3.7}$$

Here

$$\hat{u}(\omega, t) = (2\pi)^{-d/2} \int_{\mathbb{T}^d} \mathrm{e}^{-\mathrm{i}\omega\cdot x} u(x, t)\,\mathrm{d}x, \quad \omega \cdot x = \sum_j \omega_j x_j,$$

and

$$\hat{P}(\mathrm{i}\omega) = -|\omega|^2 I + \mathrm{i}\sum_{j=1}^{d} \omega_j A_j + B \tag{3.8}$$

is the symbol of P. Denoting the Laplace transform of $\hat{u}(\omega, t)$ by $\tilde{u}(\omega, s)$, we obtain from (3.7)

$$s\tilde{u} = \hat{P}(\mathrm{i}\omega)\tilde{u} + \tilde{F}. \tag{3.9}$$

As we will show, good estimates of u in terms of F can be obtained if the matrices $\hat{P}(\mathrm{i}\omega)$ satisfy the eigenvalue condition of the ODE case uniformly in ω. Accordingly, we make the following assumption, which we will discuss at the end of the section.

Assumption 3.2 (Eigenvalue Assumption) There exists $\delta > 0$ such that

$$\operatorname{Re}\lambda \le -\delta < 0 \quad \text{for all} \quad \lambda \in \sigma(\hat{P}(\mathrm{i}\omega)), \quad \omega \in \mathbb{Z}^d.$$

If Assumption 3.2 is satisfied, the matrices $sI - \hat{P}(\mathrm{i}\omega)$ are nonsingular for $\operatorname{Re} s \ge 0$, and one has the following uniform estimate for the resolvents of $\hat{P}(\mathrm{i}\omega)$.

Lemma 3.1 If Assumption 3.2 holds, then there is a constant K_1 with

$$|(sI - \hat{P}(\mathrm{i}\omega))^{-1}| \le \frac{K_1}{|\omega|^2 + 1} \quad \text{for all } \operatorname{Re} s \ge 0, \quad \omega \in \mathbb{Z}^d. \tag{3.10}$$

Proof. First consider large $|\omega|$. We write

$$\begin{aligned} sI - \hat{P}(\mathrm{i}\omega) &= \left(s + |\omega|^2\right) I - \hat{Q}(\omega) \qquad (\text{with } |\hat{Q}| \le C(|\omega| + 1)) \\ &= \left(s + |\omega|^2\right)\left(I - \mathcal{O}\left(\frac{1}{|\omega|}\right)\right). \end{aligned} \tag{3.11}$$

Thus, there is $C > 0$ such that (3.10) holds for $|\omega| \ge C$.

There are only finitely many $\omega \in \mathbb{Z}^d$ with $|\omega| \le C$. For these ω-vectors, we apply the resolvent estimate of Lemma 2.2b. □

Using (3.9) and (3.10) we can estimate $\tilde{u}$ in terms of $\tilde{F}$,

$$\left(|\omega|^2 + 1\right)|\tilde{u}(\omega, s)| \leq K_1|\tilde{F}(\omega, s)|,$$

and Parseval's relation yields the basic inequality

$$\int_0^\infty \|u(\cdot, t)\|_{H^2}^2 \, dt \leq K_2 \int_0^\infty \|F(\cdot, t)\|^2 \, dt. \tag{3.12}$$

If we first apply D^α to the differential equation $u_t = Pu + F$ and sum the resulting inequalities (3.12) over $|\alpha| \leq p$, we find

$$\int_0^\infty \|u(\cdot, t)\|_{H^{p+2}}^2 \, dt \leq K_2 \int_0^\infty \|F(\cdot, t)\|_{H^p}^2 \, dt, \quad p = 0, 1, 2, \ldots. \tag{3.13}$$

Using the differential equation $u_t = Pu + F$, we can also estimate u_t and its space derivatives. For example,

$$\|u_t\| \leq C(\|u\|_{H^2} + \|F\|)$$

since P is of second order. Furthermore, since values of $F(x, t)$ for $t > T$ do not affect the solution $u(x, t)$ for $t \leq T$, we can restrict the resulting estimates to any finite time interval. Let us summarize these results.

Theorem 3.2 Let $u_t = Pu + F$ satisfy the Eigenvalue Assumption (Assumption 3.2). There is a constant K, independent of F and T, with

$$\int_0^T \left(\|u\|_{H^{p+2}}^2 + \|u_t\|_{H^p}^2\right) dt \leq K \int_0^T \|F\|_{H^p}^2 \, dt, \quad p = 0, 1, 2, \ldots. \tag{3.14}$$

To estimate u (and some of its derivatives) in maximum norm, we will make use of the following Sobolev inequality.

Theorem 3.3 Let $v \in H^p(\mathbb{T}^d)$. If $p > \frac{d}{2}$ then $v \in C(\mathbb{T}^d)$ and

$$|v|_\infty \leq C_{p,d}\|v\|_{H^p}. \tag{3.15}$$

The constant $C_{p,d}$ does not depend on v.

By (2.29) we have, for any x,

$$\max_{0 \leq t \leq T} |u(x, t)|^2 \leq \int_0^T \left(|u(x, t)|^2 + |u_t(x, t)|^2\right) dt.$$

Taking the maximum over $x \in \mathbb{T}^d$ and using the above Sobolev inequality, we find that

$$\max_{0 \leq t \leq T} |u(\cdot, t)|_\infty^2 \leq C \int_0^T \left(\|u\|_{H^p}^2 + \|u_t\|_{H^p}^2\right) dt \quad \text{if } p > \frac{d}{2}. \tag{3.16}$$

Again, we can apply the same estimate to derivatives $D^\alpha u$ and obtain

$$\max_{0 \leq t \leq T} |D^\alpha u(\cdot, t)|_\infty^2 \leq C \int_0^T \left(\|u\|_{H^p}^2 + \|u_t\|_{H^p}^2\right) dt \quad \text{if } p > |\alpha| + \frac{d}{2}. \tag{3.17}$$

In terms of $v = (u, Du, D^2u)$, we have the following bound:

$$\max_{0\le t\le T} |D^\alpha v(\cdot,t)|_\infty^2 \le C \int_0^T \left(\|u\|_{H^p}^2 + \|u_t\|_{H^p}^2 \right) dt \quad \text{if } p > |\alpha| + 2 + \frac{d}{2}. \quad (3.18)$$

(Clearly, estimates (3.17) and (3.18) are valid for any sufficiently regular function $u(x,t)$ with $u = 0$ at $t = 0$; the PDE has not been used.)

After these preparations, let us prove Theorem 3.1.

The nonlinear problem

Consider (3.1) for any ε. There exists $T_\varepsilon > 0$ with

$$\int_0^{T_\varepsilon} \left(\|u\|_{H^{p+2}}^2 + \|u_t\|_{H^p}^2 \right) dt \le 4K \int_0^\infty \|F\|_{H^p}^2 \, dt. \quad (3.19)$$

(The consideration of (3.19) is motivated by the fact that (3.19) is valid for $\varepsilon = 0$ with $4K$ replaced by K; see Theorem 3.2.) We want to prove that (3.19) holds for $T_\varepsilon = \infty$ if $|\varepsilon|$ is sufficiently small. As before, we set

$$v = (u, Du, D^2u)$$

and use the linear estimate of Theorem 3.2 with $F(x,t)$ replaced by $F(x,t) + \varepsilon f(x,t,v(x,t))$, to obtain

$$\begin{aligned} &\int_0^{T_\varepsilon} \left(\|u\|_{H^{p+2}}^2 + \|u_t\|_{H^p}^2 \right) dt \\ &\quad \le \; 2K \int_0^\infty \|F\|_{H^p}^2 \, dt + 2K|\varepsilon|^2 \int_0^{T_\varepsilon} \|f(\cdot,t,v(\cdot,t))\|_{H^p}^2 \, dt. \quad (3.20) \end{aligned}$$

It remains to prove that we can bound the integral $\int_0^{T_\varepsilon} \|f\|_{H^p}^2 \, dt$ in terms of the left-hand side of (3.20). Basically, this turns out to be possible because the left-hand side of (3.20) dominates the maximum norm of sufficiently many derivatives of v if p is large. As we will see, our choice $p = d + 5$ suffices.

The main technical difficulty is treated in the following theorem. In its proof, the simple estimate

$$\|\phi\psi\| \le |\phi|_\infty \|\psi\|$$

for the L_2-norm of the product of two functions is used. The definition of κ in the theorem is motivated by (3.18).

Theorem 3.4 (estimate based on chain rule) Let $v : \mathbb{T}^d \to \mathbb{C}^m$ and $f : \mathbb{C}^m \to \mathbb{C}^n$ denote functions of class C^p, where $p = d + 5$. Assume

$$|v(x)| \le B, \quad x \in \mathbb{T}^d,$$

and let C_B denote a bound for the derivatives of $f(v)$ in the ball $|v| \le B$,

$$|D_v^\gamma f(v)| \le C_B \quad \text{if } |v| \le B, \; |\gamma| \le p.$$

We set

$$\kappa := \max\{|D^\alpha v|_\infty \ : \ |\alpha| + 2 + \frac{d}{2} < p\}.$$

(The maximum is taken over all multi-indices α with $|\alpha| + 2 + \frac{d}{2} < p$.) Then the composite function $f(v(x)), x \in \mathbb{T}^d$, satisfies

$$\|D^\alpha (f \circ v)\| \le C C_B \left(1 + \kappa^{p-1}\right) \|v\|_{H^p} \tag{3.21}$$

for $1 \le |\alpha| \le p$. The constant C is independent of v and f.

Proof. By the chain rule

$$D^\alpha (f \circ v)(x) = \sum_\sigma c_\sigma \phi_\sigma(v(x)) D^{\sigma_1} v \ldots D^{\sigma_k} v,$$

where $\sigma_1, \ldots, \sigma_k$ are multi-indices with

$$\sigma_1 + \cdots + \sigma_k = \alpha,$$

$\phi_\sigma(v)$ is a derivative of $f(v)$ of order $\le p$, and c_σ are numerical coefficients. Therefore,

$$\|D^\alpha (f \circ v)\| \le C_1 C_B \sum_\sigma \|D^{\sigma_1} v \ldots D^{\sigma_k} v\|.$$

A factor $D^{\sigma_j} v$ can be bounded in maximum norm by κ if

$$p > |\sigma_j| + 2 + \frac{d}{2}.$$

Suppose there are two factors, $D^{\sigma_1} v$ and $D^{\sigma_2} v$, say, whose maximum norm *cannot* be bounded by κ. Then

$$p \le |\sigma_1| + 2 + \frac{d}{2} \quad \text{and} \quad p \le |\sigma_2| + 2 + \frac{d}{2};$$

thus

$$2p \le |\sigma_1| + |\sigma_2| + 4 + d \le p + d + 4.$$

However, this contradicts our choice $p = d + 5$. Therefore, each product

$$D^{\sigma_1} v \ldots D^{\sigma_k} v$$

contains at most one factor which cannot be estimated in maximum norm by κ. We conclude

$$\|D^{\sigma_1} v \ldots D^{\sigma_k} v\| \le C \left(1 + \kappa^{p-1}\right) \|v\|_{H^p}, \quad 1 \le k \le p.$$

This proves the theorem. □

Remark 3.2 In our bound of $\|D^{\sigma_1}v\dots D^{\sigma_k}v\|$, we have only used the simple inequality $\|\phi\psi\| \le |\phi|_\infty\|\psi\|$ and Sobolev's inequality. The condition on p can be relaxed if one instead uses Hölder's inequality and a Gagliardo–Nirenberg inequality. See, for example, Hagstrom and Lorenz (1995) or Racke (1992).

Note that the estimate (3.21) does not hold, in general, for $\alpha = 0$, as the example $f \equiv 1$ shows. In our application we assume, however, that f vanishes for $v = 0$. A corresponding result is formulated next.

Theorem 3.5 Let $v : \mathbb{T}^d \to \mathbb{C}^m$ and $f : \mathbb{T}^d \times \mathbb{C}^m \to \mathbb{C}^n$ denote functions of class C^p, where $p = d + 5$. We consider the composite function

$$f(x, v(x)), \quad x \in \mathbb{T}^d.$$

Assuming that

$$|v(x)| \le B, \quad x \in \mathbb{T}^d,$$

we require the following estimates for the derivatives of f in the ball $|v| \le B$:

$$\begin{aligned} |D_x^\beta f(x,v)| &\le C_B|v| \quad \text{if } |v| \le B,\ |\beta| \le p; \\ |D_x^\beta D_v^\gamma f(x,v)| &\le C_B \quad \text{if } |v| \le B,\ |\beta| + |\gamma| \le p,\ |\gamma| \ge 1. \end{aligned}$$

Setting

$$\kappa := \max\left\{ |D^\alpha v|_\infty \ :\ |\alpha| + 2 + \frac{d}{2} < p \right\},$$

we have

$$\|f(\cdot, v(\cdot))\|_{H^p} \le CC_B\left(1 + \kappa^{p-1}\right)\|v\|_{H^p},$$

where C is independent of v and f.

Proof. For $|\alpha| \le p$, the derivative $D^\alpha f(x, v(x))$ is a sum of terms

$$D_x^\beta D_v^\gamma f(x, v(x)) D^{\sigma_1}v \dots D^{\sigma_k}v \tag{3.22}$$

where

$$|\beta| + |\gamma| \le p, \quad |\sigma_1| + \dots + |\sigma_k| \le p.$$

If $|\gamma| \ge 1$, the estimate of (3.22) proceeds as in the proof of the previous theorem. If $\gamma = 0$, the factor $D^{\sigma_1}v \dots D^{\sigma_k}v$ is empty, and we use the estimate $|D_x^\beta f(x,v)| \le C_B|v|$ to obtain

$$\|D_x^\beta f\| \le C_B\|v\|.$$

The claim follows. □

It is now easy to complete the proof of nonlinear asymptotic stability stated in Theorem 3.1. The term $\int \|f\|^2_{H^p}\,\mathrm{d}t$ on the right-hand side of (3.20) can be bounded as follows:

$$\int_0^{T_\varepsilon} \|f\|^2_{H^p}\,\mathrm{d}t \le C_1 \int_0^{T_\varepsilon} \|v\|^2_{H^p}\,\mathrm{d}t \le C_2 \int_0^{T_\varepsilon} \|u\|^2_{H^{p+2}}\,\mathrm{d}t.$$

Therefore, if

$$2K|\varepsilon|^2 C_2 \le \frac{1}{3},$$

then one obtains from (3.20)

$$\int_0^{T_\varepsilon} \left(\|u\|^2_{H^{p+2}} + \|u_t\|^2_{H^p}\right)\mathrm{d}t \le 3K \int_0^\infty \|F\|^2_{H^p}\,\mathrm{d}t. \tag{3.23}$$

Since (3.19) implies (3.23), we can conclude that (3.23) is valid for $T_\varepsilon = \infty$. Convergence $\max_x |u(x,t)| \to 0$ as $t \to \infty$ (and even for $v = (u, Du, D^2u)$ instead of u) follows from

$$\int_{t_0}^\infty \left(\|u\|^2_{H^{p+2}} + \|u_t\|^2_{H^p}\right)\mathrm{d}t \to 0 \quad \text{as} \quad t_0 \to \infty.$$

Discussion of the Eigenvalue Assumption
Since $\hat{P}(\mathrm{i}\omega) = -|\omega|^2 I + \mathcal{O}(|\omega|)$ for large $|\omega|$, it follows that the Eigenvalue Assumption (Assumption 3.2) is always satisfied for large $|\omega|$. If the Eigenvalue Assumption is violated, there exists $\omega \in \mathbb{Z}^d$ and $\phi \in \mathbb{C}^n$ with

$$\hat{P}(\mathrm{i}\omega)\phi = \lambda\phi, \quad \operatorname{Re}\lambda \ge 0, \quad \phi \ne 0.$$

If we set

$$u(x,t) = \frac{1}{\lambda+1}(\mathrm{e}^{\lambda t} - \mathrm{e}^{-t})\phi, \quad F(x,t) = \mathrm{e}^{-t}\phi,$$

then $u_t = Pu + F$, but u does not tend to zero as $t \to \infty$. This shows that the Eigenvalue Assumption is necessary for linear asymptotic stability.

A simple sufficient condition for the Eigenvalue Assumption is

$$A_j = A_j^*, \quad j = 1, \dots, d; \quad B + B^* \le -2\delta I < 0.$$

In this case, $\hat{P}(\mathrm{i}\omega) + \hat{P}^*(\mathrm{i}\omega) \le B + B^* \le -2\delta I$, which implies $\operatorname{Re}\lambda \le -\delta$ for all eigenvalues λ of $\hat{P}(\mathrm{i}\omega)$.

Obviously, $\hat{P}(\mathrm{i}\omega) = B$ for $\omega = 0$. This shows that the Eigenvalue Assumption can only be satisfied if all eigenvalues of the zero-order term B of P have negative real parts. There are cases of interest, however, where $B = 0$ or B has a nontrivial null-space. Then it may still be true that $\operatorname{Re}\lambda \le -\delta < 0$ for all $\lambda \in \sigma(\hat{P}(\mathrm{i}\omega))$ if $\omega \in \mathbb{Z}^d$ and $\omega \ne 0$. Under such a restricted eigenvalue assumption one can prove a restricted form of asymptotic stability by taking a suitable projection. See, for example, Hagstrom and Lorenz (1995) for applications.

4. General PDEs: periodic boundary conditions

In this section we consider PDEs of the form

$$u_t = Pu + \varepsilon f(x,t,u) + F(x,t), \quad x \in \mathbb{R}^d, \quad t \geq 0, \tag{4.1}$$

with initial condition

$$u(x,0) = 0, \quad x \in \mathbb{R}^d. \tag{4.2}$$

Here P is a linear constant coefficient operator, that is,

$$P = \sum_{|\nu|\leq m} A_\nu D^\nu, \quad A_\nu \in \mathbb{C}^{n\times n}, \tag{4.3}$$

$$D^\nu = D_1^{\nu_1} \dots D_d^{\nu_d}, \quad D_j = \frac{\partial}{\partial x_j}, \quad |\nu| = \nu_1 + \cdots + \nu_d. \tag{4.4}$$

As in the previous section, we will assume that $F(x,t)$ and $f(x,t,u)$ are C^∞ functions, which are 2π-periodic in each variable x_j. We seek a solution $u(x,t)$ with the same spatial periodicity. We will also discuss (4.1) with more general nonlinearities

$$f(x,t,u,Du,\dots,D^r u),$$

where $D^j u$ denotes the array of all spatial derivatives of u of order j. As before, it will be assumed that the nonlinear term $f(x,t,v)$ vanishes for $v = 0$ with

$$v = (u, Du, \dots, D^r u).$$

Our aim in this section is twofold.

(1) We want to explain the significance of the eigenvalue assumption

$$\operatorname{Re}\lambda \leq -\delta < 0 \quad \text{for all } \lambda \in \sigma(\hat{P}(\mathrm{i}\omega)), \quad \omega \in \mathbb{Z}^d, \tag{4.5}$$

for the general class of operators (4.3). In fact, as we will show, if the Cauchy problem for $u_t = Pu$ is well posed in L_2, the eigenvalue assumption (4.5) always implies a resolvent estimate leading to nonlinear asymptotic stability of (4.1).

(2) The number r of derivatives of u, which are allowed in the nonlinearity f, depends on the resolvent estimate in a simple way. On the Fourier–Laplace side, one needs a bound

$$(|\omega|^q + 1)|\tilde{u}(\omega,s)| \leq K_1|\tilde{F}(\omega,s)| \quad \text{for all } \operatorname{Re} s \geq 0, \quad \omega \in \mathbb{Z}^d, \tag{4.6}$$

with $q \geq r$. If the Cauchy problem for $u_t = Pu$ is well posed, then

(4.6) is equivalent to the eigenvalue condition

$$\operatorname{Re}\lambda \le -(|\omega|^q + 1)\delta < 0 \quad \text{for all} \quad \lambda \in \sigma(\hat{P}(\mathrm{i}\omega)), \quad \omega \in \mathbb{Z}^d. \tag{4.7}$$

The simple eigenvalue condition (4.5) leads to (4.7) with $q = 0$. If q is even and $u_t = Pu$ is a parabolic system of order q, then – by the definition of parabolicity – condition (4.7) is always satisfied for large $|\omega|$. In other words, for parabolic systems of order q, the conditions (4.5) and (4.7) are equivalent and can be checked, in principle, by computing the eigenvalues of finitely many matrices $\hat{P}(\mathrm{i}\omega)$, $\omega \in \mathbb{Z}^d$.

Well-posedness

Let us briefly review the concept of well-posedness in L_2 and first consider the linear problem $u_t = Pu$ under an initial condition $u(x,0) = u_0(x)$. We make a Fourier expansion of the initial data,

$$u_0(x) = \sum_{\omega \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\omega\cdot x} \hat{u}_0(\omega), \tag{4.8}$$

and, tentatively, of the solution,

$$u(x,t) = \sum_{\omega \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\omega\cdot x} \hat{u}(\omega, t), \tag{4.9}$$

for each $t \ge 0$. Introducing the symbol of P,

$$\hat{P}(\kappa) = \sum_{|\nu| \le m} \kappa_1^{\nu_1} \dots \kappa_d^{\nu_d} A_\nu, \quad \kappa \in \mathbb{C}^d, \tag{4.10}$$

and observing that

$$P(\mathrm{e}^{\mathrm{i}\omega\cdot x}\phi) = \hat{P}(\mathrm{i}\omega)\phi, \quad \phi \in \mathbb{C}^n, \tag{4.11}$$

we obtain formally

$$\hat{u}_t(\omega, t) = \hat{P}(\mathrm{i}\omega)\hat{u}(\omega, t), \quad \hat{u}(\omega, 0) = \hat{u}_0(\omega); \tag{4.12}$$

thus

$$\hat{u}(\omega, t) = \mathrm{e}^{\hat{P}(\mathrm{i}\omega)t} \hat{u}_0(\omega). \tag{4.13}$$

The formal process is justified for $t \ge 0$ if the matrix exponentials in (4.13) have a limited exponential growth rate, which is uniform for all $\omega \in \mathbb{Z}^d$. Well-posedness can be defined accordingly; for details, see Kreiss and Lorenz (1989), for example.

Definition 4.1 The 2π-periodic Cauchy problem for $u_t = Pu$ is well posed (in L_2) if there are real constants K and α with

$$|\mathrm{e}^{\hat{P}(\mathrm{i}\omega)t}| \le K\mathrm{e}^{\alpha t} \quad \text{for all} \quad \omega \in \mathbb{Z}^d, \ t \ge 0. \tag{4.14}$$

The L_2-inner product and norm are defined by

$$(u,v) = \int_{\mathbb{T}^d} u^*(x)v(x), \quad \|u\|^2 = (u,u), \quad u,v \in L_2(\mathbb{T}^d, \mathbb{C}^n).$$

Then, if the 2π-periodic Cauchy problem is well posed and $u_0 \in C^\infty$, the formula (4.9) gives us the solution $u(x,t)$, which is C^∞ and satisfies

$$\|u(\cdot,t)\| \le K\mathrm{e}^{\alpha t}\|u_0\|, \quad t \ge 0.$$

As usual, boundedness of the assignment $u_0 \to u(\cdot,t)$ in L_2 implies that we can obtain a generalized solution for all initial data $u_0(x)$ in L_2.

Basic resolvent estimate

The following result says that the eigenvalue assumption (4.5) implies a resolvent estimate, whenever the Cauchy problem is well posed.

Theorem 4.1 Assume that $P = \sum_{|\nu|\le m} A_\nu D^\nu$ satisfies the following two conditions.

(1) The 2π-periodic Cauchy problem for $u_t = Pu$ is well posed, that is, there are constants α and K with

$$|\mathrm{e}^{\hat{P}(\mathrm{i}\omega)t}| \le K\mathrm{e}^{\alpha t} \quad \text{for all } \omega \in \mathbb{Z}^d, \quad t \ge 0.$$

(2) There is $\delta > 0$ with

$$\mathrm{Re}\,\lambda \le -\delta < 0 \quad \text{for all } \lambda \in \sigma(\hat{P}(\mathrm{i}\omega)), \quad \omega \in \mathbb{Z}^d.$$

Then there is a constant K with

$$|(sI - \hat{P}(\mathrm{i}\omega))^{-1}| \le K \quad \text{for all } \mathrm{Re}\,s \ge 0, \quad \omega \in \mathbb{Z}^d. \tag{4.15}$$

A main tool for the proof is the Kreiss matrix theorem, which we formulate next. (See Kreiss and Lorenz (1989).)

Theorem 4.2 Let $\mathcal{F}$ denote any set of matrices $A \in \mathbb{C}^{n\times n}$, where n is fixed. Then the following conditions are equivalent.

(1) There is a constant K_1 with

$$|\mathrm{e}^{At}| \le K_1 \quad \text{for all } A \in \mathcal{F}, \ t \ge 0. \tag{4.16}$$

(2) For all $A \in \mathcal{F}$ and all $s \in \mathbb{C}$ with $\mathrm{Re}\,s > 0$ the matrix $sI - A$ is nonsingular, and there is a constant K_2 with

$$|(sI - A)^{-1}| \le \frac{K_2}{\mathrm{Re}\,s} \quad \text{for all } A \in \mathcal{F}, \ \mathrm{Re}\,s > 0. \tag{4.17}$$

(3) There are constants K_{31}, K_{32} and, for all $A \in \mathcal{F}$, there is a transformation $S = S(A) \in \mathbb{C}^{n\times n}$ with $|S| + |S^{-1}| \le K_{31}$ so that

$$SAS^{-1} = \begin{pmatrix} \lambda_1 & b_{12} & \cdots & \cdots & b_{1n} \\ 0 & \lambda_2 & b_{23} & \cdots & b_{2n} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \lambda_{n-1} & b_{n-1,n} \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{pmatrix} \tag{4.18}$$

is upper-triangular, the diagonal is ordered,

$$0 \ge \operatorname{Re}\lambda_1 \ge \cdots \ge \operatorname{Re}\lambda_n,$$

and the upper-diagonal elements satisfy

$$|b_{jk}| \le K_{32}|\operatorname{Re}\lambda_j|, \quad 1 \le j < k \le n.$$

(4) There is a positive constant K_4 and, for each $A \in \mathcal{F}$, there is a Hermitian matrix $H = H(A) \in \mathbb{C}^{n\times n}$ with

$$\frac{1}{K_4} I \le H \le K_4 I\ , \quad HA + A^*H \le 0. \tag{4.19}$$

Proof of Theorem 4.1. There are constants $\alpha \ge 0, K > 0$ with

$$|\mathrm{e}^{(\hat{P}(\mathrm{i}\omega)-\alpha I)t}| \le K \quad \text{for all } \omega \in \mathbb{Z}^d, \ \ t \ge 0.$$

By the Kreiss matrix theorem – applied to the family $\hat{P}(\mathrm{i}\omega) - \alpha I$, $\omega \in \mathbb{Z}^d$, – there is a bounded transformation $S = S(\omega)$ with

$$S(\hat{P}(\mathrm{i}\omega) - \alpha I)S^{-1} = \begin{pmatrix} \lambda_1-\alpha & b_{12} & \cdots & \cdots & b_{1n} \\ 0 & \lambda_2-\alpha & b_{23} & \cdots & b_{2n} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \lambda_{n-1}-\alpha & b_{n-1,n} \\ 0 & \cdots & \cdots & 0 & \lambda_n-\alpha \end{pmatrix}$$

and

$$|b_{jk}| \le K_{32}\ |\operatorname{Re}\lambda_j - \alpha|\ , \quad j < k.$$

Using the assumption $\operatorname{Re}\lambda_j \le -\delta < 0 \le \alpha$, we obtain that

$$\begin{aligned} |\operatorname{Re}\lambda_j - \alpha| &= |\operatorname{Re}\lambda_j| \left(1 + \frac{\alpha}{|\operatorname{Re}\lambda_j|}\right) \\ &\le 2\left(1+\frac{\alpha}{\delta}\right)\left|\operatorname{Re}\frac{\lambda_j}{2}\right| \\ &\le 2\left(1+\frac{\alpha}{\delta}\right)\left|\operatorname{Re}\lambda_j + \frac{\delta}{2}\right|. \end{aligned} \tag{4.20}$$

Therefore,

$$|b_{jk}| \leq K'_{32} \left|\operatorname{Re}\lambda_j + \frac{\delta}{2}\right|, \quad j < k. \tag{4.21}$$

Considering $S(\hat{P}(\mathrm{i}\omega)+\frac{\delta}{2}I)S^{-1}$, we obtain from the estimates (4.21) that the Kreiss matrix theorem also applies to the family $\hat{P}(\mathrm{i}\omega) + \frac{\delta}{2}I$, $\omega \in \mathbb{Z}^d$. Now we use the second characterization in Theorem 4.2 and find

$$\left|\left(\left(s - \frac{\delta}{2}\right) I - \hat{P}(\mathrm{i}\omega)\right)^{-1}\right| \leq \frac{K_2}{\operatorname{Re} s}, \quad \operatorname{Re} s > 0.$$

In particular, this implies the estimate

$$\left|\left(sI - \hat{P}(\mathrm{i}\omega)\right)^{-1}\right| \leq \frac{K_2}{\delta/2}, \quad \operatorname{Re} s \geq 0,$$

and the theorem is proved. □

Asymptotic stability of (4.1)
Consider the linear problem

$$u_t = Pu + F(x,t), \quad u(x,0) = 0,$$

and let the assumptions of Theorem 4.1 hold. Fourier expansion and Laplace transformation lead to the family of linear algebraic equations

$$(sI - \hat{P}(\mathrm{i}\omega))\tilde{u}(\omega, s) = \tilde{F}(\omega, s), \quad \operatorname{Re} s \geq 0, \quad \omega \in \mathbb{Z}^d, \tag{4.22}$$

and (4.15) yields

$$|\tilde{u}(\omega, s)| \leq K|\tilde{F}(\omega, s)|.$$

By Parseval's relation this translates into the estimate

$$\int_0^\infty \|u(\cdot,t)\|^2 \,\mathrm{d}t \leq K_1 \int_0^\infty \|F(\cdot,t)\|^2 \,\mathrm{d}t.$$

Applying this basic inequality to $D^\alpha u$ and using $u_t = Pu + F$ to estimate time derivatives, one obtains

$$\int_0^\infty \left(\|u\|^2_{H^p} + \|u_t\|^2_{H^{p-m}}\right) \mathrm{d}t \leq K_2 \int_0^\infty \|F\|^2_{H^p} \,\mathrm{d}t, \quad p = m, m+1, \ldots. \tag{4.23}$$

(Recall that m is the order of P.) If $p - m > \frac{d}{2}$ and the right-hand side of (4.23) is finite, we obtain a bound for $\sup_t |u(\cdot,t)|^2$. With the same arguments as in Section 3, this shows linear asymptotic stability.

Theorem 4.3 Let p denote the smallest integer with $p - m > \frac{d}{2}$, and assume

$$\int_0^\infty \|F(\cdot,t)\|^2_{H^p} \,\mathrm{d}t < \infty.$$

If P satisfies the conditions of Theorem 4.1, then $|u(\cdot,t)|_\infty \to 0$ as $t \to \infty$.

The extension to the nonlinear problem (4.1), where $f = f(x,t,u)$ does not depend on the derivatives of u, proceeds as before. Formally, we obtain from (4.23)

$$\begin{aligned} L^2 := \int_0^T \left(\|u\|_{H^p}^2 + \|u_t\|_{H^{p-m}}^2 \right) \mathrm{d}t \\ \leq \; 2K_2 \int_0^\infty \|F\|_{H^p}^2 \, \mathrm{d}t \; + \; 2K_2|\varepsilon|^2 \int_0^T \|f\|_{H^p}^2 \, \mathrm{d}t. \end{aligned} \tag{4.24}$$

Here $f = f(x,t,u(x,t))$. We have (compare (3.17))

$$\max_{0\leq t\leq T} |D^\alpha u(\cdot,t)|_\infty^2 \leq CL^2 \quad \text{if} \;\; p > |\alpha| + m + \frac{d}{2}. \tag{4.25}$$

We estimate $\|f\|_{H^p}$ by applying the chain rule and have to consider

$$\|D^{\sigma_1}u \ldots D^{\sigma_k}u\|, \qquad |\sigma_1| + \cdots + |\sigma_k| \leq p.$$

(See the proofs of Theorems 3.4 and 3.5.) If there are two factors, $D^{\sigma_1}u$ and $D^{\sigma_2}u$, say, which are not dominated in maximum norm by L, then

$$p \leq |\sigma_1| + m + \frac{d}{2} \quad \text{and} \quad p \leq |\sigma_2| + m + \frac{d}{2};$$

thus

$$2p \leq |\sigma_1| + |\sigma_2| + 2m + d \leq p + 2m + d.$$

Therefore, if we choose $p = 2m + d + 1$, there can be at most one factor $D^{\sigma_j}u$ that cannot be dominated in maximum norm by L, and one obtains

$$\|f(\cdot,t,u(\cdot,t))\|_{H^p} \leq C\|u(\cdot,t)\|_{H^p}.$$

By the same arguments as in Section 3 we have proved nonlinear asymptotic stability.

Theorem 4.4 Let $p = 2m + d + 1$. If $\int_0^\infty \|F\|_{H^p}^2 \, \mathrm{d}t$ is finite and $f(x,t,u)$ satisfies Assumption 3.1 (with v replaced by u), then $|u(\cdot,t)|_\infty \to 0$ as $t \to \infty$ for sufficiently small $|\varepsilon|$.

Resolvent estimate gaining derivatives
It is not difficult to generalize Theorem 4.1 as follows.

Theorem 4.5 Assume that the 2π-periodic Cauchy problem for $u_t = Pu$ is well posed and that the symbols $\hat{P}(\mathrm{i}\omega)$ satisfy the following eigenvalue condition:

$$\operatorname{Re}\lambda \leq -(|\omega|^q + 1)\delta < 0 \quad \text{for all} \;\; \lambda \in \sigma(\hat{P}(\mathrm{i}\omega)), \;\; \omega \in \mathbb{Z}^d,$$

where q is a nonnegative integer. Then there is a constant K with

$$|(sI-\hat{P}(\mathrm{i}\omega))^{-1}| \le \frac{K}{|\omega|^q+1} \quad \text{for all } \operatorname{Re} s \ge 0, \quad \omega \in \mathbb{Z}^d. \tag{4.26}$$

Proof. The proof, based on the Kreiss matrix theorem, proceeds in exactly the same way as the proof of Theorem 4.1. Just note that, instead of (4.20), we have here

$$|\operatorname{Re}\lambda_j - \alpha| \le 2\left(1+\frac{\alpha}{\delta}\right)\left|\operatorname{Re}\lambda_j + \frac{\delta}{2}(|\omega|^q+1)\right|.$$

□

For the linear problem $u_t = Pu + F$, (4.26) translates into the estimates

$$\int_0^\infty \|u\|_{H^q}^2 \,\mathrm{d}t \le K_1 \int_0^\infty \|F\|^2 \,\mathrm{d}t \tag{4.27}$$

and

$$\int_0^\infty \left(\|u\|_{H^{p+q}}^2 + \|u_t\|_{H^{p+q-m}}^2\right) \mathrm{d}t \le K_2 \int_0^\infty \|F\|_{H^p}^2 \,\mathrm{d}t, \quad p \ge m-q. \tag{4.28}$$

The left-hand side of (4.28) bounds $\sup_t |u(\cdot,t)|_\infty^2$ if $p+q-m > \frac{d}{2}$, and one obtains linear asymptotic stability.

Theorem 4.6 Let P satisfy the assumptions of Theorem 4.5 and assume $\int_0^\infty \|F\|_p^2 \,\mathrm{d}t < \infty$, where p is the smallest integer with $p+q-m > \frac{d}{2}$. Then $\lim_{t\to\infty} \max_x |u(x,t)| = 0$ if $|\varepsilon|$ is sufficiently small.

Nonlinear asymptotic stability when f depends on Du, etc.
Let us assume again that P satisfies the conditions of Theorem 4.5. Thus, in the linear estimate we gain q derivatives; see (4.27) and (4.28). Recall that m is the order of P and $m \ge q$. (This follows from (4.26) for $|\omega| \to \infty$.) Let the nonlinearity f depend on (x,t,v) where

$$v = (u, Du, \dots, D^r u).$$

We want to explain why one obtains nonlinear stability if

$$r \le q,$$

but cannot allow $r > q$, in general. Here we assume, as before, that $f(x,t,v)$ vanishes for $v = 0$. More precisely, we require Assumption 3.1 for sufficiently large p.

In order to control $f(x,t,v)$, we choose p so large that the left-hand side of (4.28) dominates $|v|_\infty^2$. Consequently, since v contains $D^r u$, we let p be so large that

$$p+q-m > r + \frac{d}{2}. \tag{4.29}$$

(Further restrictions on p will appear below.) Then we have (see (4.28))

$$\int_0^T \left(\|u\|^2_{H^{p+q}} + \|u_t\|^2_{H^{p+q-m}} \right) \mathrm{d}t$$
$$\leq 2K_2 \int_0^\infty \|F\|^2_{H^p} \,\mathrm{d}t + 2K_2|\varepsilon|^2 \int_0^T \|f\|^2_{H^p} \,\mathrm{d}t, \tag{4.30}$$

with

$$f = f(x, t, v(x, t)).$$

Denote the left-hand side of (4.30) by L^2; thus

$$\max_{0 \leq t \leq T} |D^\alpha v(\cdot, t)|^2_\infty \leq CL^2 \quad \text{if} \;\; p + q - m > |\alpha| + r + \frac{d}{2}.$$

We estimate $\|f\|_{H^p}$ by applying the chain rule (see the proofs of Theorems 3.4 and 3.5), which leads to the consideration of

$$\|D^{\sigma_1} v \ldots D^{\sigma_k} v\| \quad \text{with} \;\; |\sigma_1| + \cdots + |\sigma_k| \leq p.$$

If there are two factors, $D^{\sigma_1} v$ and $D^{\sigma_2} v$, say, which cannot be estimated in maximum norm by CL, then

$$p + q - m \leq |\sigma_1| + r + \frac{d}{2} \quad \text{and} \quad p + q - m \leq |\sigma_2| + r + \frac{d}{2};$$

thus

$$2p + 2q - 2m \leq p + 2r + d.$$

Therefore, we choose p so large that (4.29) holds and

$$p > 2(r - q) + d + 2m.$$

Under these conditions on p we have

$$\|f(\cdot, t, v(\cdot, t))\|_{H^p} \leq C\|v\|_{H^p}, \quad C = C(L). \tag{4.31}$$

Thus far, no restriction on the relation between r (the number of derivatives of u in f) and q (the number of derivatives gained in the resolvent estimate) has occured. Clearly, $\|v\|_{H^p} \approx \|u\|_{H^{p+r}}$, and therefore,

$$\int_0^T \|f\|^2_{H^p} \,\mathrm{d}t \leq C \int_0^T \|u\|^2_{H^{p+r}} \,\mathrm{d}t \tag{4.32}$$

by (4.31). If $r \leq q$ and $|\varepsilon|$ is small enough, we obtain the desired bound

$$\int_0^T \left(\|u\|^2_{H^{p+q}} + \|u_t\|^2_{H^{p+q-m}} \right) \mathrm{d}t \leq 3K_2 \int_0^\infty \|F\|^2_{H^p} \,\mathrm{d}t \tag{4.33}$$

from (4.30), and nonlinear stability follows. On the other hand, if $r > q$ and we substitute (4.32) on the right-hand side of (4.30), the new right-hand side contains higher derivatives of u than the left; then we cannot obtain a bound for u.

Let us summarize our result of nonlinear asymptotic stability.

Theorem 4.7 Consider

$$u_t = Pu + \varepsilon f(x,t,v) + F(x,t), \quad u(x,0) = 0,$$

with

$$v = (u, Du, \dots, D^r u).$$

Let P satisfy the conditions of Theorem 4.5 with $q \geq r$. Furthermore, let Assumption 3.1 hold for f and let $\int_0^\infty \|F\|^2_{H^p}\,\mathrm{d}t < \infty$, where $p = 2m+d+1$. Under these assumptions, $\lim_{t\to\infty} |v(\cdot,t)|_\infty = 0$ if $|\varepsilon|$ is sufficiently small.

Discussion

As noted above, the eigenvalue condition (4.26) is reasonable if $u_t = Pu$ is a parabolic system of order q. In this case Theorem 4.7 states that the nonlinearity may depend on all space derivatives of u of order $\leq q$.

Now consider the hyperbolic equation

$$u_t = u_x - u + F(x,t).$$

We have $\hat{P}(\mathrm{i}\omega) = \mathrm{i}\omega - 1$, and the simple eigenvalue condition (4.5) is satisfied with $\delta = -1$. However, the resolvent estimate (4.26) is only fulfilled with $q = 0$, as the choice $s = \mathrm{i}\omega$ in (4.26) shows. Therefore, by Theorem 4.7, we may add a nonlinear term $\varepsilon f(x,t,u)$, but dependency of f on u_x is not allowed.

In the next section we will treat hyperbolic problems in more generality using the Lyapunov technique. It will become clear that certain nonlinearities $\varepsilon f(x,t,u,u_x)$ still lead to asymptotic stability, though the resolvent technique fails.

5. Hyperbolic problems: periodic boundary conditions

Consider a first-order system

$$u_t = Pu + \varepsilon \sum_{j=1}^{d} B_j(u) D_j u + F(x,t), \quad x \in \mathbb{R}^d, \quad t \geq 0, \tag{5.1}$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d, \tag{5.2}$$

where P is a constant coefficient operator,

$$Pu = \sum_{j=1}^{d} A_j D_j u + Bu.$$

Our main assumption is symmetry, that is,

$$A_j = A_j^*, \quad B_j(u) = B_j^*(u), \quad j = 1, \ldots, d, \tag{5.3}$$

and negativity of the zero-order term,

$$B + B^* \leq -2\delta I < 0. \tag{5.4}$$

Because of (5.3), system (5.1) is called *symmetric hyperbolic.* The functions $B_j(u), F(x,t)$, and $u_0(x)$ are assumed to be of class C^∞, for simplicity, and $F(x,t), u_0(x)$, and $u(x,t)$ are 2π-periodic in each x_j. In addition, it will be convenient here to assume that all quantities are real.

As remarked at the end of the previous section, (5.1) cannot be treated by resolvent estimates, but, as we will see, by the Lyapunov technique.

The basic energy estimate

First consider (5.1) for $\varepsilon = 0$. As in Section 2.1, we consider the 'change in energy' of the solution:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|^2 &= \frac{\mathrm{d}}{\mathrm{d}t}(u,u) \\ &= 2(u,u_t) \\ &= 2\left(u, \sum_j A_j D_j u\right) + 2(u,Bu) + 2(u,F). \end{aligned} \tag{5.5}$$

Using the symmetry of A_j, integration by parts, and the periodic boundary conditions, one obtains

$$\begin{aligned} (u, A_j D_j u) &= (A_j u, D_j u) \\ &= -(A_j D_j u, u); \end{aligned}$$

thus

$$(u, A_j D_j u) = 0.$$

Furthermore,

$$(u, Bu) = (B^* u, u) = (u, B^* u);$$

thus

$$(u, Bu) = \frac{1}{2}(u, (B + B^*)u) \leq -\delta\|u\|^2.$$

Equation (5.5) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 &\leq -2\delta\|u\|^2 + 2\|u\|\|F\| \\ &\leq -\delta\|u\|^2 + \frac{1}{\delta}\|F\|^2, \end{aligned}$$

and we obtain the basic energy estimate

$$\begin{aligned}\|u(\cdot,t)\|^2 &\le \mathrm{e}^{-\delta t}\|u_0\|^2 + \frac{1}{\delta}\int_0^t \mathrm{e}^{-\delta(t-\tau)}\|F(\cdot,\tau)\|^2\,\mathrm{d}\tau \\ &\le \mathrm{e}^{-\delta t}\|u_0\|^2 + \frac{1}{\delta^2}\max_{0\le\tau\le t}\|F(\cdot,\tau)\|^2. \end{aligned} \tag{5.6}$$

This estimate is completely analogous to (2.7).

Clearly, from $u_t = Pu + F$ we find $D^\alpha u = PD^\alpha u + D^\alpha F$, and summing the resulting estimates over all α with $|\alpha| \le p$, we obtain

$$\|u(\cdot,t)\|_{H^p}^2 \le \mathrm{e}^{-\delta t}\|u_0\|_{H^p}^2 + \frac{1}{\delta^2}\max_{0\le\tau\le t}\|F(\cdot,\tau)\|_{H^p}^2, \quad p = 0, 1, \ldots. \tag{5.7}$$

By the Sobolev inequality stated in Theorem 3.3, we can bound $|u(\cdot,t)|_\infty$ by $\|u(\cdot,t)\|_{H^p}$ if $p > \frac{d}{2}$. Therefore, arguing exactly as in the ODE case in Section 2.1, we have proved the following result of linear asymptotic stability.

Theorem 5.1 Let $P = \sum_j A_j D_j + B$ satisfy the assumptions $A_j = A_j^*$ and $B + B^* \le -2\delta I < 0$. Furthermore, assume $\lim_{t\to\infty}\|F(\cdot,t)\|_{H^p} = 0$ where p is the smallest integer with $p > \frac{d}{2}$. Then we have $\lim_{t\to\infty}|u(\cdot,t)|_\infty = 0$.

For the linear problem, we could also have used the resolvent technique if $\int_0^\infty \|F\|_{H^p}^2\,\mathrm{d}t$ were finite.

The nonlinear problem

For the solution of (5.1) we consider again the 'change in energy',

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 &= 2(u, Pu) + 2(u, F) + 2\varepsilon\sum_j (u, B_j(u)D_j u) \\ &\le -\delta\|u\|^2 + \frac{1}{\delta}\|F\|^2 + 2|\varepsilon|\sum_j |(u, B_j(u)D_j u)|. \end{aligned} \tag{5.8}$$

Using the symmetry of $B_j(u)$, integration by parts, and the periodic boundary conditions, we obtain

$$\begin{aligned}(u, B_j(u)D_j u) &= (B_j(u)u, D_j u) \\ &= -(B_j(u)D_j u, u) - (B_j'(u)(D_j u)u, u); \end{aligned}$$

thus

$$(u, B_j(u)D_j u) = -\frac{1}{2}(B_j'(u)(D_j u)u, u). \tag{5.9}$$

Here

$$|B_j'(u(\cdot,t))|_\infty \le C_1(1 + |u|_\infty)$$

and one finds

$$|(u, B_j(u)D_j u)| \le C_1(1 + |u|_\infty)|D_j u|_\infty\|u\|^2. \tag{5.10}$$

We can substitute this estimate into (5.8), but the resulting inequality does not lead to a bound for $\|u\|$, because $|u|_\infty$ and $|D_j u|_\infty$ cannot be bounded in terms of $\|u\|$.

To obtain an inequality that closes, we consider $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^p}^2$ for sufficiently large p. As we will see below, the choice $p = d+2$, which we now make, is sufficient. For $|\alpha| \le p$, we apply D^α to the differential equation (5.1) and obtain

$$D^\alpha u_t = PD^\alpha u + D^\alpha F + \varepsilon \sum_j D^\alpha(B_j D_j u).$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|D^\alpha u\|^2 \le -\delta\|D^\alpha u\|^2 + \frac{1}{\delta}\|D^\alpha F\|^2 + 2|\varepsilon| \sum_j |(D^\alpha u, D^\alpha(B_j D_j u))| . \quad (5.11)$$

By Leibnitz's rule,

$$D^\alpha(B_j D_j u) = \sum_{\beta+\gamma=\alpha} c_{\alpha\beta}(D^\beta B_j)(D^\gamma D_j u)$$

with numerical coefficients $c_{\alpha\beta}$. The most 'dangerous' term occurs for $\beta = 0, \gamma = \alpha$. On the right-hand side of (5.11), this term contributes

$$(D^\alpha u, B_j D^\alpha D_j u),$$

and, if $|\alpha| = p$, then $p+1$ derivatives are applied to u. However, using the symmetry of B_j and integration by parts, we can remove one derivative and find, as in (5.10),

$$|(D^\alpha u, B_j D^\alpha D_j u)| \le C_1(1+|u|_\infty)|D_j u|_\infty \|D^\alpha u\|^2. \quad (5.12)$$

Now let

$$|\beta| \ge 1,\ \beta+\gamma = \alpha, \quad \text{thus} \quad |\gamma| \le p-1,$$

and consider

$$\Big|(D^\alpha u, (D^\beta B_j)(D^\gamma D_j u))\Big| \le \|D^\alpha u\| \|(D^\beta B_j)(D^\gamma D_j u)\|. \quad (5.13)$$

Just as in the proof of Theorem 3.4, we apply the chain rule to write $D^\beta(B_j(u(x,t)))$ as a sum. Then one finds that $(D^\beta B_j)(D^\gamma D_j u)$ is a sum of terms

$$(D_u^\nu B_j) D^{\sigma_1} u \ldots D^{\sigma_k} u, \quad 2 \le k \le p+1,$$

where

$$|\sigma_1| + \cdots + |\sigma_k| \le p+1 \quad \text{and} \quad |\sigma_j| \le p, \quad j = 1, \ldots, k.$$

Here $D_u^\nu B_j$ is a derivative of $B_j(u)$, and thus

$$\Big|(D_u^\nu B_j)(u(\cdot,t))\Big|_\infty \le C(1+|u|_\infty). \quad (5.14)$$

It remains to bound

$$\|D^{\sigma_1}u \dots D^{\sigma_k}u\|.$$

By Sobolev's inequality,

$$|D^{\sigma_j}u|_\infty \le C\|u\|_{H^p} \quad \text{if} \quad p > |\sigma| + \frac{d}{2}.$$

Suppose there exist two factors, $D^{\sigma_1}u$ and $D^{\sigma_2}u$, say, which *cannot* be bounded in maximum norm by $\|u\|_{H^p}$. Then we would have

$$p \le |\sigma_1| + \frac{d}{2} \quad \text{and} \quad p \le |\sigma_2| + \frac{d}{2},$$

and thus

$$2p \le |\sigma_1| + |\sigma_2| + d \le p + 1 + d,$$

in contradiction to our choice $p = d+2$. We conclude that each factor $D^{\sigma_j}u$, with at most one exception, can be bounded in maximum norm by $\|u\|_{H^p}$. This implies

$$\|D^{\sigma_1}u \dots D^{\sigma_k}u\| \le C\left(1 + \|u\|_{H^p}^{p-1}\right)\|u\|_{H^p}^2, \tag{5.15}$$

since $2 \le k \le p+1$. To summarize, the inequalities (5.12), (5.13), (5.14), and (5.15) yield

$$|(D^\alpha u, D^\alpha(B_j D_j u))| \le C\left(1 + \|u\|_{H^p}^{p+1}\right)\|u\|_{H^p}^2.$$

We substitute this bound into the right-hand side of (5.11) and sum over all α with $|\alpha| \le p$ to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^p}^2 \le -\delta\|u\|_{H^p}^2 + \frac{1}{\delta}\|F\|_{H^p}^2 + 2|\varepsilon| C\left(1 + \|u\|_{H^p}^{p+1}\right)\|u\|_{H^p}^2. \tag{5.16}$$

The constant C is independent of ε and t; it depends on the size of B_j and its derivatives.

Using the differential inequality (5.16) and elementary ODE arguments, one obtains the following result of nonlinear asymptotic stability.

Theorem 5.2 Consider (5.1) under the assumptions (5.3) and (5.4). If $\lim_{t\to\infty}\|F(\cdot,t)\|_{H^p} = 0$ for $p = d+2$, then $\lim_{t\to\infty}\|u(\cdot,t)\|_{H^p} = 0$ for sufficiently small $|\varepsilon|$. In particular, $|u(\cdot,t)|_\infty \to 0$ as $t \to \infty$.

Generalizations

It is straightforward to generalize Theorem 5.2 to the case

$$B_j = B_j(x,t,u)$$

as long as one has symmetry $A_j = A_j^*$, $B_j = B_j^*$, $j = 1, \dots, d$. Also, without difficulty, a zero-order term $\varepsilon f(x,t,u)$ with $f(x,t,0) = 0$ can be included in (5.1); more precisely, Assumption 3.1 is required.

The arguments become much more involved if the symmetry assumption is dropped. Let us explain the difficulty. We start with the linear case, and assume that

$$u_t = Pu \quad \text{with} \quad Pu = \sum_j A_j D_j u + Bu,$$

is strongly hyperbolic; that is, for all $\omega \in \mathbb{R}^d$ the eigenvalues of $\sum_j \omega_j A_j$ are real and semi-simple, and there is a transformation $S = S(\omega)$ with

$$|S(\omega)| + |S^{-1}(\omega)| \leq \text{const}$$

so that

$$S\left(\sum_j \omega_j A_j\right) S^{-1}$$

is diagonal. If one assumes, in addition, the eigenvalue condition

$$\operatorname{Re}\lambda \leq -\delta < 0 \quad \text{for all} \quad \lambda \in \sigma(\hat{P}(\mathrm{i}\omega)), \quad \omega \in \mathbb{Z}^d, \tag{5.17}$$

then one can use the characterization (4) in the Kreiss matrix theorem (Theorem 4.2) and construct matrices $H = H(\omega)$ with the following properties:

$$0 < \frac{1}{C} I \leq H(\omega) = H^*(\omega) \leq CI;$$
$$H(\omega)\hat{P}(\omega) + \hat{P}^*(\omega)H(\omega) \leq -\delta H(\omega).$$

Using the matrices $H(\omega)$, which form a so-called symmetrizer, one defines a new inner product on $L_2 = L_2(\mathbb{T}^d, \mathbb{C}^n)$ by

$$(u,v)_{\mathcal{H}} = \sum_{\omega \in \mathbb{Z}^d} \hat{u}^*(\omega) H(\omega) \hat{v}(\omega).$$

The $\mathcal{H}$-inner product is equivalent to the L_2-inner product, and the operator P becomes negative in the sense that

$$2(u, Pu)_{\mathcal{H}} \leq -\delta \|u\|^2_{\mathcal{H}}.$$

(This is the main point of the construction.) For solutions of the linear equation $u_t = Pu + F$, one then obtains without difficulty

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathcal{H}} \leq -\delta \|u\|^2_{\mathcal{H}} + 2\|u\|_{\mathcal{H}} \|F\|_{\mathcal{H}}$$

and can derive a satisfactory energy estimate.

However, to treat a nonlinear equation

$$u_t = Pu + \varepsilon \sum_j B_j(u) D_j u + F(x,t),$$

the construction is not fine enough, even if $B_j(u) = B_j^*(u)$. The difficulty is that the rule

$$(D^\alpha u, B_j D^\alpha D_j u) = (B_j D^\alpha u, D^\alpha D_j u) \tag{5.18}$$

is not valid if the L_2-inner product is replaced by the $\mathcal{H}$-inner product, and a rule like (5.18) – together with integration by parts – is needed to remove a derivative from the 'dangerous' term $D^\alpha D_j u$. For this reason it is necessary, in general, to refine the construction of $H(\omega)$ by terms of order ε and to construct a symmetrizer adjusted to the solution of the nonlinear problem. Details of the construction, which uses elementary properties of pseudodifferential operators, are carried out in Kreiss, Kreiss and Lorenz (1998*b*) and Kreiss, Ortiz and Reula (1998*c*). It is assumed that either the unperturbed system is strictly hyperbolic or that the eigenvalues of the full symbol have constant multiplicities.

6. Parabolic problems in bounded domains

Model problem and basic estimate
Consider the parabolic equation

$$u_t = Pu + \varepsilon f(x,t,u,u_x,u_{xx}) + F(x,t), \quad 0 \le x \le 1, \quad t \ge 0, \tag{6.1}$$

where P is the second-order operator

$$Pu = u_{xx} + a(x)u_x + b(x)u.$$

We require the initial and boundary conditions

$$u(x,0) = 0, \quad 0 \le x \le 1; \quad u(0,t) = u_x(1,t) = 0, \quad t \ge 0. \tag{6.2}$$

The given scalar functions $a(x), b(x), F(x,t)$ and $f(x,t,u,u_x,u_{xx})$ are assumed to be of class C^∞, for simplicity, and compatibility of the data with the boundary conditions is assumed. The functions f, f_t, f_{tt}, and f_{ttt} are required to satisfy Assumption 3.1 with $v = (u, u_x, u_{xx})$ for all sufficiently large p.

For $\varepsilon = 0$, Laplace transformation leads to a family of ordinary BVPs for $\tilde{u} = \tilde{u}(x,s)$, namely

$$s\tilde{u} = \tilde{u}_{xx} + a(x)\tilde{u}_x + b(x)\tilde{u} + \tilde{F}(x,s), \quad 0 \le x \le 1, \tag{6.3}$$

$$\tilde{u}(0,s) = \tilde{u}_x(1,s) = 0. \tag{6.4}$$

A basic observation is that one can always obtain good estimates of $\tilde{u}$ by $\tilde{F}$, gaining two derivatives, if $\operatorname{Re} s \ge 0$ and $|s|$ is sufficiently large.

Lemma 6.1 There are constants C and K, depending only on $|a|_\infty + |b|_\infty$, so that the solution of (6.3), (6.4) satisfies

$$|s|^2\|\tilde{u}\|^2 + |s|\|\tilde{u}_x\|^2 + \|\tilde{u}_{xx}\|^2 \le K\|\tilde{F}\|^2 \tag{6.5}$$

if $\operatorname{Re} s \ge 0$ and $|s| \ge C$.

Proof. Take the L_2-inner product of (6.3) with $\tilde{u}(x,s)$ and use integration by parts to obtain

$$s\|\tilde{u}\|^2 + \|\tilde{u}_x\|^2 = (\tilde{u}, a\tilde{u}_x) + (\tilde{u}, b\tilde{u}) + (\tilde{u}, \tilde{F}) =: R. \tag{6.6}$$

The absolute value of the right-hand side is bounded by

$$\begin{aligned} |R| &\leq |a|_\infty \|\tilde{u}\|\|\tilde{u}_x\| + |b|_\infty \|\tilde{u}\|^2 + \|\tilde{u}\|\|\tilde{F}\| \\ &\leq \frac{1}{2}\|\tilde{u}_x\|^2 + K_1\|\tilde{u}\|^2 + \|\tilde{u}\|\|\tilde{F}\|. \end{aligned}$$

Taking the real part of (6.6), we find

$$\operatorname{Re} s\|\tilde{u}\|^2 + \frac{1}{2}\|\tilde{u}_x\|^2 \leq K_1\|\tilde{u}\|^2 + \|\tilde{u}\|\|\tilde{F}\|. \tag{6.7}$$

Case 1: $\operatorname{Re} s \geq |\operatorname{Im} s|$; thus $|s| \leq \sqrt{2}\operatorname{Re} s$.
If $K_1 \leq \frac{|s|}{2\sqrt{2}}$, we obtain from (6.7)

$$\frac{|s|}{2\sqrt{2}}\|\tilde{u}\|^2 + \frac{1}{2}\|\tilde{u}_x\|^2 \leq \|\tilde{u}\|\|\tilde{F}\| \leq \frac{|s|}{4\sqrt{2}}\|\tilde{u}\|^2 + \frac{\sqrt{2}}{|s|}\|\tilde{F}\|^2;$$

thus

$$|s|^2\|\tilde{u}\|^2 + |s|\|\tilde{u}_x\|^2 \leq 8\|\tilde{F}\|^2. \tag{6.8}$$

Case 2: $0 \leq \operatorname{Re} s \leq |\operatorname{Im} s|$, thus $|s| \leq \sqrt{2}|\operatorname{Im} s|$.
First, from (6.7) and $\operatorname{Re} s \geq 0$ we find

$$\frac{1}{2}\|\tilde{u}_x\|^2 \leq K_1\|\tilde{u}\|^2 + \|\tilde{u}\|\|\tilde{F}\|. \tag{6.9}$$

Also, taking the imaginary part of (6.6), we have

$$|\operatorname{Im} s|\|\tilde{u}\|^2 \leq K_2\Big(\|\tilde{u}\|^2 + \|\tilde{u}_x\|^2\Big) + \|\tilde{u}\|\|\tilde{F}\|,$$

and, together with (6.9), we obtain

$$|\operatorname{Im} s|\|\tilde{u}\|^2 \leq K_3\Big(\|\tilde{u}\|^2 + \|\tilde{u}\|\|\tilde{F}\|\Big).$$

Recalling that $|s| \leq \sqrt{2}|\operatorname{Im} s|$, we obtain, as before,

$$|s|^2\|\tilde{u}\|^2 \leq K_4\|\tilde{F}\|^2 \quad \text{for} \quad |s| \geq C,$$

if C is sufficiently large. Together with (6.9), we have shown that

$$|s|^2\|\tilde{u}\|^2 + |s|\|\tilde{u}_x\|^2 \leq K_5\|\tilde{F}\|^2 \quad \text{for} \quad |s| \geq C. \tag{6.10}$$

Since we have proved such an estimate already in Case 1 (see (6.8)), it is clear that (6.10) is generally valid for $\operatorname{Re} s \geq 0, |s| \geq C$. Finally, using the differential equation (6.3), we can estimate $\|\tilde{u}_{xx}\|$ by $K_5(|s|\|\tilde{u}\| + \|\tilde{u}_x\| + \|\tilde{F}\|)$, and the lemma is proved. □

A simple implication of Lemma 6.1 is the unique solvability of the BVP

$$s\phi = P\phi + g(x), \qquad \phi(0) = \phi_x(1) = 0,$$

where

$$P\phi = \phi_{xx} + a\phi_x + b\phi,$$

provided that $\operatorname{Re} s \geq 0, |s| \geq C$. Here $g(x)$ is any inhomogeneous term. In particular, it follows that the eigenvalue problem

$$P\phi = \lambda\phi, \qquad \phi(0) = \phi_x(1) = 0 \tag{6.11}$$

does not have an eigenvalue λ with $\operatorname{Re}\lambda \geq 0, |\lambda| \geq C$.

To obtain asymptotic stability, we formulate the following eigenvalue condition.

Assumption 6.1 The eigenvalue problem (6.11) has no eigenvalue λ with $\operatorname{Re}\lambda \geq 0$.

Remark 6.1 Lemma 6.1 excludes *large* eigenvalues λ with nonnegative real part but, depending on $a(x)$ and $b(x)$, eigenvalues λ with $\operatorname{Re}\lambda \geq 0$ and $|\lambda| \leq C$ can exist, of course. Since one only has to examine a compact λ-region, Assumption 6.1 can be tested with standard numerical procedures. Also, sufficient conditions for Assumption 6.1 are well known from the maximum principle. For example, if $a(x)$ and $b(x)$ are real and $b(x) \leq 0$ for all x or $b(x) - a_x(x) \leq 0$ for all x, then Assumption 6.1 holds.

Assumption 6.1 together with Lemma 6.1 gives us a strong resolvent estimate.

Theorem 6.1 Consider

$$P\phi = \phi_{xx} + a(x)\phi_x + b(x)\phi, \quad 0 \leq x \leq 1.$$

Then Assumption 6.1 holds if and only if there is a constant K such that

$$s\tilde{u} = P\tilde{u} + \tilde{F}, \quad \tilde{u}(0,s) = \tilde{u}_x(1,s), \quad \operatorname{Re} s \geq 0, \tag{6.12}$$

implies

$$\|\tilde{u}\|_{H^2} \leq K\|\tilde{F}\|. \tag{6.13}$$

Proof. First, assuming that the estimate (6.13) holds for all $\operatorname{Re} s \geq 0$, there can be no eigenvalue λ with $\operatorname{Re}\lambda \geq 0$; that is, Assumption 6.1 holds.

Second, assume that the eigenvalue problem (6.11) has no eigenvalue λ with $\operatorname{Re}\lambda \geq 0$. Then, for every fixed s with $\operatorname{Re} s \geq 0$, the BVP (6.12) has a unique solution, and the estimate (6.13) holds with $K = K(s)$. Furthermore, if s varies in a compact region, the constants $K(s)$ can be chosen uniformly bounded, as one can prove by a contradiction argument and Arcela's theorem. Therefore, Lemma 6.1 completes the proof. □

By Parseval's relation (with $\operatorname{Re} s = 0$), (6.13) translates into

$$\int_0^\infty \|u(\cdot,t)\|_{H^2}^2 \, \mathrm{d}t \le K_1 \int_0^\infty \|F(\cdot,t)\|^2 \, \mathrm{d}t. \tag{6.14}$$

Estimates for derivatives and linear stability
In the case of periodic boundary conditions, we could apply an estimate like (6.14) directly to $D^\alpha u$ since $D^\alpha u_t = PD^\alpha u + D^\alpha F$. However, in the present case, the boundary conditions $u(0,t) = u_x(1,t) = 0$ have been used to derive (6.14), and $D^\alpha u$ does not satisfy these boundary conditions, in general. Instead, we differentiate with respect to t to obtain

$$u_{tt} = Pu_t + F_t.$$

Let us assume that

$$u_t(x,0) = 0, \quad 0 \le x \le 1. \tag{6.15}$$

We will show below that this is no restriction. Then we can apply (6.14) to u_t and find

$$\int_0^\infty \|u_t\|_{H^2}^2 \, \mathrm{d}t \le K_1 \int_0^\infty \|F_t\|^2 \, \mathrm{d}t. \tag{6.16}$$

Further, if $u_{tt}(x,0) \equiv 0$, then we can repeat the process and find

$$\int_0^\infty \|u_{tt}\|_{H^2}^2 \, \mathrm{d}t \le K_1 \int_0^\infty \|F_{tt}\|^2 \, \mathrm{d}t. \tag{6.17}$$

Since

$$u_{txx} = u_{xxxx} + (au_x)_{xx} + (bu)_{xx} + F_{xx},$$

it is not difficult to show that (6.14) and (6.16) imply

$$\int_0^\infty \left(\|u\|_{H^4}^2 + \|u_t\|_{H^2}^2\right) \mathrm{d}t \le K_2 \int_0^\infty \left(\|F\|_{H^2}^2 + \|F_t\|^2\right) \mathrm{d}t. \tag{6.18}$$

The Sobolev inequality, which we stated in Theorem 3.3 for periodic functions, remains valid without periodicity and, therefore, the estimate (3.17) applies here. Consequently, the left-hand side of (6.18) dominates

$$\sup_t \left(|u(\cdot,t)|_\infty^2 + |u_x(\cdot,t)|_\infty^2\right).$$

Using the same arguments as in Section 3, we obtain $\lim_{t\to\infty} |u(\cdot,t)|_\infty = 0$ if the right hand side of (6.18) is finite.

It remains to show that the assumption (6.15) is not restrictive. To this end, consider (6.1), (6.2) and make the change of variables

$$u(x,t) = t\mathrm{e}^{-t}\phi(x) + v(x,t),$$

where $\phi(x)$ will be determined. At $t = 0$ we have

$$F(x,0) = u_t(x,0) = \phi(x) + v_t(x,0).$$

Therefore, if we choose $\phi(x) = F(x,0)$ then we obtain $v_t(x,0) = 0$ for the new variable.

Nonlinear stability

For illustration, let us assume first that the nonlinearity f in (6.1) has the form $f = f(u_x)$. Proceeding as in Section 3, we consider (see (6.18))

$$\begin{aligned}\int_0^T \left(\|u\|_{H^4}^2 + \|u_t\|_{H^2}^2\right) dt \qquad\qquad (6.19)\\ \leq \; 2K_2 \int_0^\infty \left(\|F\|_{H^2}^2 + \|F_t\|^2\right) dt + 2K_2|\varepsilon|^2 \int_0^T \left(\|f\|_{H^2}^2 + \|(f)_t\|^2\right) dt.\end{aligned}$$

Here

$$f = f(u_x(x,t)), \qquad (f)_t = f'(u_x(x,t))u_{xt}(x,t).$$

As noted above, the left-hand side of (6.19) dominates $\max_{0 \leq t \leq T} |u_x(\cdot,t)|_\infty^2$ and, therefore, the nonlinearity is controlled in maximum norm. Furthermore,

$$\int_0^T \|u_{xt}\|^2 \, dt$$

is also bounded by the left-hand side of (6.19). Using the same arguments as in Section 3, we find that $\lim_{t\to\infty} |u(\cdot,t)|_\infty = 0$ if the right-hand side of (6.19) is finite and $|\varepsilon|$ is sufficiently small.

In the general case $f = f(x,t,u,u_x,u_{xx})$ we proceed similarly. After proper initialization, we have the following generalization of (6.18), restricted to a finite time interval:

$$\begin{aligned} L^2 \; &:= \; \int_0^T \left(\|u\|_{H^8}^2 + \|u_t\|_{H^6}^2 + \|u_{tt}\|_{H^4}^2 + \|u_{ttt}\|_{H^2}^2\right) dt \\ &\leq \; K_3 \int_0^T \left(\|F\|_{H^6}^2 + \|F_t\|_{H^4}^2 + \|F_{tt}\|_{H^2}^2 + \|F_{ttt}\|^2\right) dt,\end{aligned}$$

The left-hand side bounds $\sup_t |D^\alpha u|_\infty^2$ for $\alpha \leq 5$. (Here $D = \partial/\partial x$.) To treat the nonlinear problem, we need to bound

$$\int_0^T \left(\|f\|_{H^6}^2 + \|(f)_t\|_{H^4}^2 + \|(f)_{tt}\|_{H^2}^2 + \|(f)_{ttt}\|^2\right) dt$$

in terms of L^2. Applying the chain rule and expressing any t-derivative on u by two x-derivatives (using the differential equation), one needs to consider

$$\|D^{\sigma_1}u \ldots D^{\sigma_k}u\|, \quad \sigma_1 + \cdots + \sigma_k \leq 8.$$

Since derivatives up to order 5 are controlled in maximum norm by L, one finds that

$$\|D^{\sigma_1}u \dots D^{\sigma_k}u\|^2 \leq C\|u\|^2_{H^8}.$$

The remaining arguments are as in Section 3.

Generalizations

It is not difficult to generalize the key result, Lemma 6.1, to the Laplace transforms of parabolic systems

$$u_t = (A(x)u_x)_x + B(x)u_x + C(x)u + F(x,t) \equiv Pu + F(x,t)$$

under initial and boundary conditions

$$u(x,0) = 0, \quad 0 \leq x \leq 1; \quad R_0 u(0,t) = R_1 u(1,t) = 0, \quad t \geq 0.$$

Here $A(x), B(x)$, and $C(x)$ are smooth matrix functions, and parabolicity requires

$$A(x) = A^*(x) \geq \alpha I > 0.$$

We also assume that the boundary conditions $R_0\tilde{u} = R_1\tilde{u} = 0$ imply

$$\langle \tilde{u}, A\tilde{u}_x \rangle \Big|_0^1 = 0.$$

(This boundary term appears when $(\tilde{u}, (A\tilde{u}_x)_x)$ is integrated by parts.) In particular, one can use a Dirichlet or Neumann condition. Under these assumptions, one obtains that a strong resolvent estimate

$$\|\tilde{u}\|_{H^2} \leq K\|\tilde{F}\| \quad \text{for all } \operatorname{Re} s \geq 0$$

holds if and only if the eigenvalue problem

$$P\phi = \lambda\phi, \qquad R_0\phi = R_1\phi = 0,$$

has no eigenvalue λ with $\operatorname{Re}\lambda \geq 0$. The arguments are the same as in the proof of Theorem 6.1. Again, the eigenvalue condition can be tested numerically since the existence of large eigenvalues λ with $\operatorname{Re}\lambda \geq 0$ is excluded by analytical arguments.

7. PDEs on all space with negative zero-order term

The purpose of this section is to extend the results of Sections 3, 4 and 5 from the case of periodic boundary conditions to problems on all space. The extension is easy, because the constant coefficient operator P is assumed to have a 'negative' zero-order term. This allows us to obtain a resolvent estimate that is valid uniformly up to $\operatorname{Re} s = 0$. For hyperbolic problems, the zero-order term leads to exponential decay. Equations without such a zero-order term are treated in Section 8.

7.1. Problems on all space with strong resolvent estimate

Consider a Cauchy problem

$$u_t = Pu + \varepsilon f(x,t,u,Du,\dots,D^r u) + F(x,t), \quad x \in \mathbb{R}^d, t \geq 0 \tag{7.1}$$

with homogeneous initial condition

$$u(x,0) = 0, \quad x \in \mathbb{R}^d. \tag{7.2}$$

As in Section 4, the operator P has constant coefficients

$$P = \sum_{|\nu|\leq m} A_\nu D^\nu, \quad A_\nu \in \mathbb{C}^{n\times n}. \tag{7.3}$$

The functions $F(x,t)$ and $f(x,t,v)$ with

$$v = (u, Du, \dots, D^r u)$$

are assumed to be of class C^∞, for simplicity. Furthermore, let $f(x,t,0) = 0$; more precisely, we require Assumption 3.1 with a sufficiently large p. For the function $F(x,t)$ we assume[8]

$$\|D^\alpha F(\cdot,t)\| < \infty, \quad \text{for all } \alpha, \quad t \geq 0,$$

and

$$\int_0^\infty \|F(\cdot,t)\|^2_{H^p}\,\mathrm{d}t < \infty$$

for a sufficiently large p.

We will always assume that the Cauchy problem $u_t = Pu, u(x,0) = u_0(x)$, is well posed in L_2, that is, there are constants K and α with

$$\left|\mathrm{e}^{\hat{P}(\mathrm{i}\omega)t}\right| \leq K\mathrm{e}^{\alpha t} \quad \text{for all } \omega \in \mathbb{R}^d, \quad t \geq 0.$$

(See, for example, Kreiss and Lorenz (1989) for a discussion of well-posedness in L_2.)

For $\varepsilon = 0$, Fourier transformation in x and Laplace transformation in t yield the family of linear algebraic equations

$$s\tilde{u}(\omega,s) = \hat{P}(\mathrm{i}\omega)\tilde{u}(\omega,s) + \hat{F}(\omega,s). \tag{7.4}$$

Here the Fourier–Laplace transform of u is

$$\tilde{u}(\omega,s) = (2\pi)^{-d/2}\int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-st-\mathrm{i}\omega\cdot x} u(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

As in Section 4, an eigenvalue condition for the symbols $\hat{P}(\mathrm{i}\omega)$ leads to a strong resolvent estimate.

[8] The L_2-inner product and norm are now defined by $(u,v) = \int_{\mathbb{R}^d} u^*(x)v(v)\,\mathrm{d}x$, $\|u\|^2 = (u,u)$, *i.e.*, the domain of integration is $\mathbb{R}^d$ instead of $\mathbb{T}^d$.

Theorem 7.1 Assume that the Cauchy problem for $u_t = Pu$ is well posed and that there are constants $q \in \{0, 1, \ldots\}$ and $\delta > 0$ with

$$\operatorname{Re}\lambda \leq -\Big(|\omega|^q + 1\Big)\delta < 0 \quad \text{for all} \quad \lambda \in \sigma(\hat{P}(\mathrm{i}\omega)), \quad \omega \in \mathbb{R}^d.$$

Then there is a constant K with

$$\left|\Big(sI - \hat{P}(\mathrm{i}\omega)\Big)^{-1}\right| \leq \frac{K}{|\omega|^q + 1} \quad \text{for all} \quad \omega \in \mathbb{R}^d, \quad \operatorname{Re}s \geq 0. \tag{7.5}$$

Proof. The proof, based on the Kreiss matrix theorem, is the same as the proof of Theorem 4.5. The essential argument is given in the proof of Theorem 4.1. One only has to replace $\omega \in \mathbb{Z}^d$ by $\omega \in \mathbb{R}^d$. □

Given that P satisfies the assumptions of Theorem 7.1, one obtains from (7.4)

$$|\tilde{u}(\omega, s)| \leq \frac{K}{|\omega|^q + 1}|\tilde{F}(\omega, s)| \quad \text{for all} \quad \omega \in \mathbb{R}^d, \quad \operatorname{Re}s \geq 0. \tag{7.6}$$

Then Parseval's relation (with $\operatorname{Re}s = 0$) yields

$$\int_0^\infty \|u(\cdot, t)\|_{H^q}^2 \, \mathrm{d}t \leq K_1 \int_0^\infty \|F(\cdot, t)\|^2 \, \mathrm{d}t.$$

We can apply this estimate to $D^\alpha u$ and can also obtain bounds for u_t and $D^\alpha u_t$ using the differential equation $u_t = Pu + F$. Therefore,

$$\begin{aligned} \int_0^\infty \Big(\|u(\cdot, t)\|_{H^{p+q}}^2 + \|u_t(\cdot, t)\|_{H^{p+q-m}}^2\Big) \, \mathrm{d}t \\ \leq \; K_2 \int_0^\infty \|F(\cdot, t)\|_{H^p}^2 \, \mathrm{d}t, \quad p \geq m - q. \end{aligned} \tag{7.7}$$

(Here m is the order of P.) The Sobolev inequality, which we formulated in Theorem 3.3 for periodic functions, is also valid for $u \in H^p(\mathbb{R}^d)$. Consequently, the left-hand side of (7.7) dominates

$$\sup_t |D^\alpha u(\cdot, t)|_\infty^2 \quad \text{if} \quad p + q - m > |\alpha| + \frac{d}{2}.$$

In exactly the same way as we have proved Theorem 4.7, we obtain the following result.

Theorem 7.2 Consider the problem (7.1), (7.2) and assume that P satisfies the conditions of Theorem 7.1 with $q \geq r$. Furthermore, let Assumption 3.1 hold for f and let $\int_0^\infty \|F\|_{H^p}^2 \, \mathrm{d}t < \infty$, where $p = 2m + d + 1$. Under these assumptions $\lim_{t\to\infty} |v(\cdot, t)|_\infty = 0$ if $|\varepsilon|$ is sufficiently small. (Here $v = (u, Du, \ldots, D^r u)$.)

7.2. Hyperbolic problems on all space

Consider a first-order system

$$u_t = Pu + \varepsilon \sum_{j=1}^{d} B_j(u) D_j u + F(x,t), \quad x \in \mathbb{R}^d, \ \ t \geq 0, \tag{7.8}$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d. \tag{7.9}$$

Here P has constant coefficients,

$$Pu = \sum_{j=1}^{d} A_j D_j u + Bu.$$

We assume symmetry,

$$A_j = A_j^*, \quad B_j(u) = B_j^*(u), \quad j = 1, \ldots, d, \tag{7.10}$$

and negativity of the zero-order term,

$$B + B^* \leq -2\delta I < 0. \tag{7.11}$$

The functions $B_j(u), F(x,t)$, and $u_0(x)$ are assumed to be of class C^∞ and, for convenience, all quantities are assumed to be real. Furthermore, let

$$\|u_0\|_{H^p} < \infty, \quad \|F(\cdot,t)\|_{H^p} < \infty \quad \text{for all } \ p = 0,1,\ldots \quad \text{and all } \ t \geq 0. \tag{7.12}$$

Under these assumptions, one knows local (in time) existence of a C^∞ solutions $u(x,t)$, and this solution satisfies

$$\|u(\cdot,t)\|_{H^p} < \infty, \quad p = 0,1,\ldots. \tag{7.13}$$

in its interval of existence. (See, for example, Kreiss and Lorenz (1989).)

To discuss stability, we first let $\varepsilon = 0$. Consider the 'change in energy',

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 &= (u, u_t) \\ &= \sum_j (u, A_j D_j u) + (u, Bu) + (u, F). \end{aligned}$$

Using the symmetry of A_j and integration by parts, one finds that

$$(u, A_j D_j u) = (A_j u, D_j u) = -(A_j D_j u, u),$$

and therefore $(u, A_j D_j u) = 0$. When one integrates by parts, boundary terms appear. However, these terms are zero, since the solution u decays to zero for $|x| \to \infty$. This follows from (7.13). For this reason, all arguments used in the spatially periodic case in Section 5 can be used here in the same way. Instead of Theorem 5.2 one obtains the following result.

Theorem 7.3 Consider the symmetric hyperbolic system (7.1) with initial condition (7.2) under the assumptions described above. Also, for $p = d + 2$ we assume $\int_0^\infty \|F\|_{H^p}^2 \, dt < \infty$. Then we have $\lim_{t\to\infty} |u(\cdot, t)|_\infty = 0$ if $|\varepsilon|$ is sufficiently small.

The generalizations outlined at the end of Section 5 can also be made in the all-space case. The only difference is that one has to work with Fourier integrals instead of Fourier sums. In particular, if $H(\omega)$ denotes a bounded symmetrizer satisfying

$$H(\omega)\hat{P}(\mathrm{i}\omega) + \hat{P}^*(\mathrm{i}\omega)H(\omega) \leq -\delta H(\omega),$$

then the $\mathcal{H}$-inner product becomes

$$(u, v)_{\mathcal{H}} = \int_{\mathbb{R}^d} \hat{u}^*(\omega)H(\omega)v(\omega)\,\mathrm{d}\omega.$$

With respect to this inner product, the linear operator P is negative:

$$(u, Pu)_{\mathcal{H}} + (Pu, u)_{\mathcal{H}} \leq -\delta\|u\|_{\mathcal{H}}^2.$$

Perturbed hyperbolic systems

$$u_t = Pu + \varepsilon \sum_j B_j(x, t, u)D_j u + F(x, t),$$

which are either strictly hyperbolic or whose full symbol

$$\sum_j \omega_j A_j + \varepsilon \sum_j \omega_j B(_j(x, t, u), \quad |\omega| = 1,$$

has eigenvalues with constant multiplicities, can again be treated by constructing a norm adjusted to the solution.

8. Parabolic problems on all space with weak resolvent estimate

In this section we consider viscous conservation laws of the form

$$u_t = Pu + \varepsilon_1 P_1 u + \varepsilon_2 \sum_{j=1}^d D_j f_j(u) + \sum_{j=1}^d D_j F_j(x, t), \quad x \in \mathbb{R}^d, \quad t \geq 0, \tag{8.1}$$

with homogeneous initial conditions

$$u(x, 0) = 0, \quad x \in \mathbb{R}^d.$$

Here, the operator P has constant coefficients, that is,

$$Pu = \Delta + \sum_{j=1}^d A_j D_j u, \quad A_j \in \mathbb{R}^{n\times n}. \tag{8.2}$$

The term $\varepsilon_1 P_1 u$ describes linear perturbations with variable coefficients, so that,

$$P_1 u = \sum_{j=1}^{d} D_j(B_j(x,t)u), \tag{8.3}$$

and the nonlinear functions $f_j(u)$ vanish quadratically at $u = 0$. Note that all terms on the right-hand side of (8.1) are derivative terms, that is, (8.1) has conservation form. In particular, the constant coefficient operator P does not have a negative zero-order term, and therefore the results of Section 7 do not apply here. The aim of the section is to show that the resolvent technique can still be used, but one needs more specific assumptions about the form of the perturbation terms. Let us list our assumptions for the terms $B_j(x,t), f_j(u)$, and $F_j(x,t)$ appearing in (8.1).

Assumption 8.1

(1) $F_j(x,t), B_j(x,t)$, and $f_j(u)$ are of class C^∞ and

$$f_j(0) = 0, \quad Df_j(0) = 0;$$

(2)
$$\int_0^\infty \int_{\mathbb{R}^d} |F(x,t)|\, dx\, dt < \infty;$$

(3)
$$\int_0^\infty \|F(\cdot,t)\|_{H^p}^2\, dt < \infty, \quad p = 0,1,\ldots;$$

(4)
$$\int_0^\infty \|B(\cdot,t)\|^2\, dt < \infty;$$

(5)
$$\sup_{x,t} |D^\alpha B(x,t)| < \infty \quad \text{for all } \alpha.$$

Our standard form (8.1) together with these assumptions and homogeneous initial conditions might seem very restrictive, but one can often enforce the requirements by simple transformations. Let us illustrate this.

8.1. Transformation to standard form

Consider a system

$$u_t = Pu + \sum_{j=1}^{d} D_j f_j(u) \tag{8.4}$$

of viscous conservation laws, where P has the form (8.2) and where the flux functions $f_j : \mathbb{R}^n \to \mathbb{R}^n$ are of class C^∞ and vanish quadratically at $u = 0$. Clearly, $u \equiv 0$ is a solution of (8.4), and we are interested in its asymptotic stability in the sense of Lyapunov; that is, we consider (8.4) with small initial data

$$u(x,0) = \varepsilon U_0(x), \quad x \in \mathbb{R}^d. \tag{8.5}$$

Conditions on $U_0(x)$ will be derived below. For simplicity, let us assume that the flux functions $f_j(u)$ are quadratic; that is, there are symmetric bilinear functions $Q_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with

$$f_j(u) = Q_j(u,u).$$

(If $H_{jk} \in \mathbb{R}^{n\times n}$ is the Hessian of the kth component of f_j at $u = 0$, then $(Q_j(u,v))_k = \frac{1}{2}v^t H_{jk} u$.) If we write $u(x,t) = \varepsilon v(x,t)$, then (8.4), (8.5) becomes

$$v_t = Pv + \varepsilon \sum_{j=1}^{d} D_j f_j(v), \quad v(x,0) = U_0(x). \tag{8.6}$$

To derive estimates by Laplace transformation, it will be convenient to initialize first. To this end, we introduce a new variable $w(x,t)$ by

$$v(x,t) = \mathrm{e}^{-t} U_0(x) + w(x,t),$$

for which we obtain

$$w(x,0) = 0. \tag{8.7}$$

Setting

$$\bar{v}(x,t) = \mathrm{e}^{-t} U_0(x)$$

we find that the function $w(x,t)$ satisfies

$$w_t = Pw + \varepsilon \sum_{j=1}^{d} D_j f_j(\bar{v} + w) + P\bar{v} - v_t,$$

where

$$f_j(\bar{v} + w) = f_j(\bar{v}) + f_j(w) + 2Q_j(\bar{v}, w).$$

Thus we can write

$$w_t = Pw + \varepsilon \sum_{j=1}^{d} D_j(B_j(x,t)w) + \varepsilon \sum_{j=1}^{d} D_j f_j(w) + G(x,t)$$

with

$$G = P\bar{v} - \bar{v}_t + \varepsilon \sum_{j=1}^{d} D_j f_j(\bar{v})$$

and matrices $B_j(x,t)$ determined by

$$B_j(x,t)w = 2Q_j(\bar{v}(x,t), w).$$

If we now assume that the initial function $U_0(x)$ has the form

$$U_0(x) = \sum_{j=1}^{d} D_j U_{0j}(x),$$

where $U_{0j} \in C^\infty$ and

$$D^\alpha U_{0j} \in L_1 \cap L_\infty \quad \text{for all} \quad \alpha,$$

then our construction shows that the inhomogeneous term $G(x,t)$ can be written as

$$G(x,t) = \sum_{j=1}^{d} D_j F_j(x,t),$$

where $F_j \in C^\infty$ and $D^\alpha F_j \in L_1 \cap L_\infty$ for all α. Therefore, $w(x,t)$ solves equation (8.1). Also, it is not difficult to show that Assumption 8.1 is satisfied.

8.2. Estimates for the unperturbed problem

Consider the linear equation

$$u_t = Pu + \sum_{j=1}^{d} D_j F_j(x,t), \quad x \in \mathbb{R}^d, \quad t \geq 0, \tag{8.8}$$

with initial condition

$$u(x,0) = 0, \quad x \in \mathbb{R}^d. \tag{8.9}$$

Here $P = \Delta + \sum_j A_j D_j$, and the $F_j(x,t)$ satisfy the relevant conditions of Assumption 8.1. We also require the following.

Assumption 8.2 The system $u_t = \sum_j A_j D_j u$ is strongly hyperbolic; that is, for all $\omega \in \mathbb{R}^d$ the eigenvalues of $\sum_j \omega_j A_j$ are real and semi-simple, and there is a transformation $S = S(\omega)$ with

$$|S(\omega)| + |S^{-1}(\omega)| \leq \text{const}$$

so that

$$S\left(\sum_j \omega_j A_j\right) S^{-1} =: \Lambda(\omega)$$

is diagonal.

Fourier–Laplace transformation of (8.8) yields

$$s\tilde{u}(\omega,s) = \hat{P}(\mathrm{i}\omega)\tilde{u}(\omega,s) + \mathrm{i}\sum_j \omega_j \tilde{F}_j(\omega,s), \tag{8.10}$$

with

$$\hat{P}(\mathrm{i}\omega) = -|\omega|^2 + \mathrm{i}\sum_j \omega_j A_j.$$

The following technical lemma contains a crucial estimate of the resolvent of $\hat{P}(\mathrm{i}\omega)$.

Lemma 8.1 There is a constant C_1, independent of $\omega \in \mathbb{R}^d$ and $\eta \geq 0$, with

$$\int_{-\infty}^{\infty} \left| \left((\eta + \mathrm{i}\xi)I - \hat{P}(\mathrm{i}\omega) \right)^{-1} \right|^2 \mathrm{d}\xi \leq C_1 |\omega|^{-2}. \tag{8.11}$$

Proof. Let $s = \eta + \mathrm{i}\xi, \eta \geq 0$. Using the transformation $S = S(\omega)$ of Assumption 8.2, we have

$$S(sI - \hat{P})S^{-1} = \left(s + |\omega|^2\right) I - \mathrm{i}\Lambda,$$

where $\Lambda = \mathrm{diag}(\lambda_k), \lambda_k \in \mathbb{R}$. Therefore,

$$\left|(sI - \hat{P})^{-1}\right|^2 \leq C \sum_{k=1}^{n} \frac{1}{(\eta + |\omega|^2)^2 + (\xi - \lambda_k)^2}. \tag{8.12}$$

Clearly, for $\eta \geq 0$ and $\omega \neq 0$,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{(\eta + |\omega|^2)^2 + (\xi - \lambda_k)^2} \leq \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{|\omega|^4 + \xi^2} = \pi|\omega|^{-2}.$$

This proves the lemma. □

We use the abbreviation

$$M(F,T) = \left(\int_0^T \int_{\mathbb{R}^d} |F(x,t)| \, \mathrm{d}x \, \mathrm{d}t \right)^2$$

for the square of the L_1-norm of F over space and the time interval $0 \leq t \leq T$. Recall that $M(F, \infty)$ is finite by Assumption 8.1. From the definition of the Fourier–Laplace transform, we obtain directly

$$|\tilde{F}(\omega, s)|^2 \leq M(F, \infty) \quad \text{for all } \ \omega \in \mathbb{R}^d, \quad \mathrm{Re}\, s \geq 0. \tag{8.13}$$

Therefore, using (8.10),

$$\begin{aligned} |\tilde{u}(\omega, s)|^2 &\leq \left|(sI - \hat{P})^{-1}\right|^2 |\omega|^2 \left|\tilde{F}(\omega, s)\right|^2 \\ &\leq M(F, \infty)|\omega|^2 \left|(sI - \hat{P})^{-1}\right|^2. \end{aligned} \tag{8.14}$$

We now apply Parseval's relation (see (2.21)) with $\eta = 0$ to obtain

$$\begin{aligned} \int_0^{\infty} |\hat{u}(\omega, t)|^2 \, \mathrm{d}t &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{u}(\omega, \mathrm{i}\xi)|^2 \, \mathrm{d}\xi \\ &\leq C_2 M(F, \infty), \quad \omega \in \mathbb{R}^d. \end{aligned} \tag{8.15}$$

In the last estimate we have used (8.14) and (8.11). The bound (8.15) will give us the crucial estimate for the small-ω projection of the solution.

Definition 8.1 Let $u = u(x), x \in \mathbb{R}^d$, denote an L_2-function with Fourier transform $\hat{u}(\omega)$. Let $\hat{u}(\omega) = \hat{u}^I(\omega) + \hat{u}^{II}(\omega)$, where

$$\hat{u}^I(\omega) = \hat{u}(\omega) \quad \text{for} \quad |\omega| \leq 1, \qquad \hat{u}^I(\omega) = 0 \quad \text{for} \quad |\omega| > 1,$$

and let $u^I(x)$ and $u^{II}(x)$ denote the corresponding inverse Fourier transforms. We call u^I and u^{II} the small-ω and the large-ω projections of u, respectively.

A similar notation, $u^{I,II}(x,t)$ and $\tilde{u}^{I,II}(\omega,s)$, will be used for functions $u(x,t)$ and their Fourier–Laplace transforms. Note that the time variable t and the dual variable s are irrelevant for the projections. Also, if $u(x,t)$ is a smooth function with derivatives in L_2, then differentiation and projection commute, because differentiation corresponds to multiplication on the Fourier side, which clearly commutes with cut-off. In particular, one obtains

$$\left|(D^\alpha u^I)^{\sim}(\omega,s)\right| \leq |\omega|^{|\alpha|}|\tilde{u}^I(\omega,s)| \leq |\tilde{u}^I(\omega,s)|. \tag{8.16}$$

Theorem 8.1 Let $u(x,t)$ solve (8.8), (8.9), and recall Assumptions 8.1 and 8.2. For any $p = 0, 1, \ldots$, there exists C_p, independent of F and T, with

$$\int_0^T \left(\|u\|^2_{H^{p+1}} + \|u_t\|^2_{H^{p-1}}\right) \mathrm{d}t \leq C_p \left(M(F,T) + \int_0^T \|F\|^2_{H^p}\, \mathrm{d}t\right). \tag{8.17}$$

Proof.

(1) We first estimate u^I. By Parseval's relation we have

$$\|u^I(\cdot,t)\|^2 = \int_{|\omega|\leq 1} |\hat{u}(\omega,t)|^2\, \mathrm{d}\omega.$$

Integrating this equation in time and observing (8.15), we find

$$\int_0^\infty \|u^I(\cdot,t)\|^2\, \mathrm{d}t \leq CM(F,\infty).$$

By (8.16) we obtain the same estimate for every derivative $D^\alpha u^I$. Therefore,

$$\int_0^\infty \|u^I\|^2_{H^p}\, \mathrm{d}t \leq C_p M(F,\infty), \quad p = 0, 1, \ldots. \tag{8.18}$$

(2) The estimate of the large-ω projection u^{II} proceeds like the estimates in Sections 3 to 5. First note that (8.12) implies

$$(|\omega|^2 + 1)\left|\left(sI - \hat{P}(\mathrm{i}\omega)\right)^{-1}\right| \leq C, \quad |\omega| \geq 1, \quad \eta \geq 0.$$

Therefore,

$$\int_0^\infty \|u^{II}\|^2_{H^{p+1}}\,\mathrm{d}t \le C_p \int_0^\infty \|F\|^2_{H^p}\,\mathrm{d}t, \quad p = 0, 1, \dots. \tag{8.19}$$

(Note that we only gain one derivative, because F appears in differentiated form as forcing.) Together with (8.18) we have derived

$$\int_0^\infty \|u\|^2_{H^{p+1}}\,\mathrm{d}t \le C_p \left(M(F,\infty) + \int_0^\infty \|F\|^2_{H^p}\,\mathrm{d}t \right), \quad p = 0, 1, \dots. \tag{8.20}$$

To estimate time derivatives, we use the differential equation. Clearly,

$$D^\alpha u_t = P D^\alpha u + \sum_j D_j D^\alpha F_j$$

yields

$$\|u_t\|^2_{H^{p-1}} \le C\Big(\|u\|^2_{H^{p+1}} + \|F\|^2_{H^p} \Big).$$

Together with (8.20), the estimate (8.17) follows for $T = \infty$. Since values of $F(x,t)$ for $t > T$ do not affect the solution $u(x,t)$ for $t \le T$, it follows that (8.17) also holds for all finite T. □

Remark 8.1 Suppose the inhomogeneous term in (8.8) is a general function

$$G \in L_1(\mathbb{R}^d \times [0,\infty)),$$

that is, we do not assume the structure $G = \sum_j D_j F_j$. Then we still have

$$|\tilde{G}(\omega, s)|^2 \le M(G,\infty) < \infty, \quad \omega \in \mathbb{R}^d, \quad \operatorname{Re} s \ge 0.$$

Therefore, $\tilde{u} = (sI - \hat{P})^{-1}\tilde{G}$ and Lemma 8.1 yield

$$\begin{aligned} \int_0^\infty |\hat{u}(\omega,t)|^2\,\mathrm{d}t &= \frac{1}{2\pi} \int_{-\infty}^\infty |\tilde{u}(\omega,\xi)|^2\,\mathrm{d}\xi \\ &\le CM(G,\infty)|\omega|^{-2}. \end{aligned} \tag{8.21}$$

In one and two space dimensions,

$$\int_{|\omega|\le 1} |\omega|^{-2}\,\mathrm{d}\omega = \infty,$$

and consequently we cannnot obtain a bound of $\int_0^\infty \|u^I\|^2\,\mathrm{d}t$ from (8.21).

However, if the number of space dimensions is $d \geq 3$, then

$$\int_{|\omega|\leq 1} |\omega|^{-2}\,\mathrm{d}\omega$$

is finite, and (8.21) yields

$$\int_0^\infty \|u^I\|^2\,\mathrm{d}t \leq C_1 M(G,\infty).$$

For this reason, the structural assumptions on the forcing and the perturbation terms in (8.1) can be relaxed for $d \geq 3$.

By (3.16), the left-hand side of (8.17) dominates $\max_{0\leq t\leq T} |u(\cdot,t)|_\infty^2$ if $p-1 > \frac{d}{2}$. With the same arguments as in Section 3, linear asymptotic stability follows.

Theorem 8.2 Consider (8.8), (8.9) under Assumptions 8.1 and 8.2. Then we have $\lim_{t\to\infty} |u(\cdot,t)| = 0$.

8.3. Stability for the perturbed problem

Consider (8.1) with initial condition $u(x,0)=0$ and recall Assumptions 8.1 and 8.2. We want to show asymptotic stability, if $\varepsilon_1^2+\varepsilon_2^2$ is small enough. The basic idea is the same as in Section 3: We use the linear estimate of Theorem 8.1 with F replaced by

$$F(x,t) + \varepsilon_1 \sum_j B_j(x,t)u(x,t) + \varepsilon_2 \sum_j f_j(u(x,t)), \tag{8.22}$$

that is, we treat the perturbation terms as forcings.

To be specific, let $p = d+3$ and let C_p be fixed with (8.17). For all $\varepsilon_1, \varepsilon_2$ there exists $T = T(\varepsilon_1,\varepsilon_2)$ so that

$$\begin{aligned} L^2 &:= \int_0^T \left(\|u\|_{H^{p+1}}^2 + \|u_t\|_{H^{p-1}}^2\right)\mathrm{d}t \\ &\leq 4C_p\left(M(F,\infty) + \int_0^\infty \|F\|_{H^p}^2\,\mathrm{d}t\right) =: R^2. \end{aligned} \tag{8.23}$$

By (3.17), we have

$$\max_{0\leq t\leq T} |D^\alpha u(\cdot,t)|_\infty^2 \leq L^2 \quad \text{if } p-1 > |\alpha| + \frac{d}{2}.$$

From (8.17) (with F replaced by (8.22)) we find

$$\begin{aligned} &\int_0^T \left(\|u\|_{H^{p+1}}^2 + \|u_t\|_{H^{p-1}}^2\right)\mathrm{d}t \\ &\quad\leq C_p\left(M(F+\varepsilon_1 Bu + \varepsilon_2 f(u),T) + \int_0^T \|F+\varepsilon_1 Bu+\varepsilon_2 f(u)\|_{H^p}^2\,\mathrm{d}t\right). \end{aligned} \tag{8.24}$$

Here

$$M(F+\varepsilon_1 Bu+\varepsilon_2 f(u),T)\le 2M(F,T)+4\varepsilon_1^2 M(Bu,T)+4\varepsilon_2 M(f(u),T)$$

and

$$\|F+\varepsilon_1 Bu+\varepsilon_2 f(u)\|_{H^p}^2\le 2\|F\|_{H^p}^2+4\varepsilon_1^2\|Bu\|_{H^p}^2+4\varepsilon_2^2\|f(u)\|_{H^p}^2,$$

and it remains to show that the quantities

$$M(Bu,T),\quad M(f(u),T),\quad \int_0^T\|Bu\|_{H^p}^2,\quad \int_0^T\|f(u)\|_{H^p}^2$$

can be estimated by $K_R L^2$, with a constant K_R depending only on the (fixed) right-hand side of (8.23). Firstly,

$$\begin{aligned} M(Bu,T) &\le \left(\int_0^T\int_{\mathbb{R}^d}|B(x,t)||u(x,t)|\,\mathrm{d}x\,\mathrm{d}t\right)^2\\ &\le \int_0^T\int_{\mathbb{R}^d}|B|^2\,\mathrm{d}x\,\mathrm{d}t\int_0^T\|u\|^2\,\mathrm{d}t\\ &\le C_B L^2, \end{aligned}$$

where we have used Assumption 8.1. Secondly, since f vanishes quadratically at $u=0$ and $|u(x,t)|\le L\le R$, we have $|f(u)|\le C_R|u|^2$. Therefore,

$$\begin{aligned} M(f(u),T) &= \left(\int_0^T\int_{\mathbb{R}^d}|f(u(x,t))|\,\mathrm{d}x\,\mathrm{d}t\right)^2\\ &\le C_R^2\left(\int_0^T\int_{\mathbb{R}^d}|u|^2\,\mathrm{d}x\,\mathrm{d}t\right)^2\\ &\le C_R^2L^4\le C_R'L^2. \end{aligned}$$

Thirdly, by Leibnitz's rule,

$$D^\alpha(B_j u)=\sum_{\beta\le\alpha}c_{\alpha\beta}(D^{\alpha-\beta}B_j)(D^\beta u);$$

thus

$$\|Bu\|_{H^p}^2\le C_B\|u\|_{H^p}^2$$

if we observe Assumption 8.1. This implies

$$\int_0^T\|Bu\|_{H^p}^2\,\mathrm{d}t\le C_B L^2.$$

Finally, a bound

$$\int_0^T \|f(u)\|_{H^p}^2 \, dt \le C(R)L^2$$

is shown exactly as in Section 3. (See Theorems 3.4 and 3.5.) These arguments prove that there is a constant K_R, independent of T and ε_1 and ε_2, such that

$$\begin{aligned} L^2 := & \int_0^T \left(\|u\|_{H^{p+1}}^2 + \|u_t\|_{H^{p-1}}^2\right) dt \\ \le & \; 2C_p \left(M(F,\infty) + \int_0^\infty \|F\|_{H^p}^2 \, dt\right) + K_R L^2 \left(\varepsilon_1^2 + \varepsilon_2^2\right). \end{aligned} \quad (8.25)$$

If we choose $\varepsilon_1^2 + \varepsilon_2^2$ so small that

$$K_R \left(\varepsilon_1^2 + \varepsilon_2^2\right) \le \frac{1}{3},$$

then (8.25) implies

$$\int_0^T \left(\|u\|_{H^{p+1}}^2 + \|u_t\|_{H^{p-1}}^2\right) dt \le 3C_p \left(M(F,\infty) + \int_0^\infty \|F\|_{H^p}^2 \, dt\right).$$

The remaining arguments are as in Section 3.

Theorem 8.3 Consider (8.1) with initial condition $u(x,0) = 0$ and recall Assumptions 8.1, 8.2. If $\varepsilon_1^2 + \varepsilon_2^2$ is small enough, then $\lim_{t\to\infty} |u(\cdot,t)|_\infty = 0$.

9. Half-space problems with strong resolvent estimate

Let $\mathcal{H}^d$ denote the half-space

$$\mathcal{H}^d = \{x = (x_1, x_2, \ldots, x_d) : \; x \in \mathbb{R}^d, x_1 \ge 0\}$$

with boundary

$$\partial\mathcal{H}^d = \{x \in \mathcal{H}^d : \; x_1 = 0\}.$$

As a model problem, we consider the scalar parabolic equation

$$\begin{aligned} u_t &= \Delta u + \sum_{j=1}^d a_j D_j u + bu + G(x,t) \\ &=: Pu + G(x,t), \quad x \in \mathcal{H}^d, \quad t \ge 0, \end{aligned} \quad (9.1)$$

with initial condition

$$u(x,0) = 0, \quad x \in \mathcal{H}^d, \quad (9.2)$$

and Dirichlet boundary condition

$$u(x,t) = 0, \quad x \in \partial\mathcal{H}^d, \quad t \ge 0. \quad (9.3)$$

In (9.1) the coefficients $a_1, \dots, a_d$, and b are real constants, and $G(x,t)$ is a smooth function, which decays sufficiently fast for $|x| \to \infty$ and for $t \to \infty$. Also, it will be assumed that $G(x,t)$ and the initial and boundary conditions are compatible at $(x,t) = (0,0)$. It is sufficient, but not necessary, that G is of class C^∞ and has compact support in $\mathcal{H}^d \times (0,\infty)$.

As in the previous sections, we will derive estimates of u in terms of the inhomogeneous term G. Then the form of the estimate will tell which type of nonlinear perturbation can be added. As we will see, the stability of the problem (9.1), (9.2), (9.3) depends crucially on the sign of b. If $b > 0$, the problem is unstable. If $b < 0$, the case treated in this section, one can derive a strong resolvent estimate. The case $b = 0$ is more intricate and is treated in the next section.

We remark that the maximum principle can be applied to the scalar equation (9.1) and also to nonlinear perturbations of it. While this principle is very useful for scalar problems, it does not apply to systems of equations in a natural way. In contrast, the approach we present here can be generalized to systems.

Proceeding formally, we Fourier transform in the tangential variables

$$x_- = (x_2, \dots, x_d)$$

and Laplace transform in t to derive a family of ordinary BVPs on the half-line $0 \le x_1 < \infty$. Using the notation

$$\tilde{u}(x_1, \omega, s) = (2\pi)^{-(d-1)/2} \int_0^\infty \int_{\mathbb{R}^{d-1}} \mathrm{e}^{-st - \mathrm{i}\omega \cdot x_-} u(x_1, x_-, t)\, \mathrm{d}x_-\, \mathrm{d}t,$$
$$\omega = (\omega_2, \dots, \omega_d) \in \mathbb{R}^{d-1}, \quad \operatorname{Re} s \ge 0,$$

we obtain

$$s\tilde{u} = D_1^2 \tilde{u} - |\omega|^2 \tilde{u} + a_1 D_1 \tilde{u} + \mathrm{i} \sum_{j=2}^{d} \omega_j a_j \tilde{u} + b\tilde{u} + \tilde{G}(x_1, \omega, s),$$

that is,

$$D_1^2 \tilde{u} + a_1 D_1 \tilde{u} - \sigma \tilde{u} = -\tilde{G}(x_1, \omega, s) \tag{9.4}$$

with

$$\sigma = s + |\omega|^2 - \mathrm{i}\rho - b, \quad \rho = \sum_{j=2}^{d} \omega_j a_j. \tag{9.5}$$

The boundary condition (9.3) transforms to

$$\tilde{u}(0, \omega, s) = 0, \quad \omega \in \mathbb{R}^{d-1}, \quad \operatorname{Re} s \ge 0. \tag{9.6}$$

In Section 9.1 we present some elementary results on ordinary BVPs of the type (9.4), (9.6).

9.1. Auxiliary results on BVPs on the half-line

To simplify notation, we write x instead of x_1, a instead a_1, u instead of $\tilde{u}$, and $F(x)$ instead of $-\tilde{G}(x_1, \omega, s)$. Then the BVP (9.4), (9.6) reads

$$u_{xx} + au_x - \sigma u = F(x), \quad 0 \le x < \infty; \quad u(0) = 0. \tag{9.7}$$

Before discussing (9.7), we look at even simpler first-order equations.

Lemma 9.1 Let $\operatorname{Re}\lambda < 0$ and consider

$$u_x = \lambda u + F(x), \quad 0 \le x < \infty,$$

where[9] $F \in C \cap L_2$. The general solution

$$u(x) = \mathrm{e}^{\lambda x} u(0) + \int_0^x \mathrm{e}^{\lambda(x-\xi)} F(\xi)\,\mathrm{d}\xi \tag{9.8}$$

satisfies the estimate

$$\|u\|^2 \le C_\lambda \Big(|u(0)|^2 + \|F\|^2 \Big)$$

with a constant C_λ independent of $u(0)$ and F.

Proof. For $u_1(x) = \mathrm{e}^{\lambda x} u(0)$ we have

$$\|u_1\|^2 = |u(0)|^2 \int_0^\infty \mathrm{e}^{2\operatorname{Re}\lambda x}\,\mathrm{d}x = \frac{|u(0)|^2}{2|\operatorname{Re}\lambda|}.$$

Let $u_2(x)$ denote the integral term in the general solution. Applying the Cauchy–Schwartz inequality, we find

$$\begin{aligned} |u_2(x)|^2 &\le \left(\int_0^x \mathrm{e}^{\frac{1}{2}\operatorname{Re}\lambda(x-\xi)} \mathrm{e}^{\frac{1}{2}\operatorname{Re}\lambda(x-\xi)} |F(\xi)|\,\mathrm{d}\xi \right)^2 \\ &\le \frac{1}{|\operatorname{Re}\lambda|} \int_0^x \mathrm{e}^{\operatorname{Re}\lambda(x-\xi)} |F(\xi)|^2\,\mathrm{d}\xi. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u_2\|^2 &\le \frac{1}{|\operatorname{Re}\lambda|} \int_0^\infty \int_\xi^\infty \mathrm{e}^{\operatorname{Re}\lambda(x-\xi)} |F(\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi \\ &= \frac{1}{|\operatorname{Re}\lambda|^2} \|F\|^2. \end{aligned}$$

The estimate of $\|u\|^2$ follows from $\|u\|^2 \le 2\|u_1\|^2 + 2\|u_2\|^2$. $\square$

[9] The condition of continuity, $F \in C$, can be dropped here and in the following if one works with weak derivatives of u instead of classical derivatives.

Lemma 9.2 Let $\operatorname{Re}\lambda > 0$ and consider

$$u_x = \lambda u + F(x), \quad 0 \le x < \infty,$$

where $F \in C \cap L_2$. There is a unique solution in L_2, namely

$$u(x) = -\int_x^\infty \mathrm{e}^{\lambda(x-\xi)} F(\xi)\, \mathrm{d}\xi. \tag{9.9}$$

This solution satisfies

$$\|u\| \le \frac{1}{\operatorname{Re}\lambda}\|F\|, \qquad |u(0)|^2 \le \frac{1}{2\operatorname{Re}\lambda}\|F\|^2.$$

Proof. If u is given by (9.9), then the estimate of $\|u\|$ follows as in the proof of the previous lemma. Also, the bound for $|u(0)|$ follows from the Cauchy–Schwartz inequality.

The general solution has an additional term $\mathrm{e}^{\lambda x}c$, but it is clear that (9.9) is the only solution in L_2. □

In the discussion of (9.7), the roots $\lambda_{1,2}$ of the characteristic equation

$$\lambda^2 + a\lambda - \sigma = 0 \tag{9.10}$$

will be important. We show the following elementary result.

Lemma 9.3 Let $a \in \mathbb{R}$ and let $\operatorname{Re}\sigma \ge 0, \sigma \ne 0$. Then the roots of (9.10) satisfy

$$\operatorname{Re}\lambda_1 < 0 < \operatorname{Re}\lambda_2.$$

Proof. Clearly, $\lambda_1 + \lambda_2 = -a$, $\lambda_1\lambda_2 = -\sigma$. Suppose a root $\lambda_1 = \mathrm{i}\alpha$ is purely imaginary. Then $\alpha \ne 0$ since $\sigma \ne 0$. However, $\lambda_2 = -a - \mathrm{i}\alpha$, thus

$$\operatorname{Re}\lambda_1\lambda_2 = \alpha^2 = -\operatorname{Re}\sigma \le 0.$$

This contradiction shows that no root can be purely imaginary.

Now let

$$\lambda_1 = x_1 + \mathrm{i}\alpha, \quad \lambda_2 = x_2 - \mathrm{i}\alpha, \quad x_j, \alpha \in \mathbb{R}.$$

Then

$$\operatorname{Re}\lambda_1\lambda_2 = x_1x_2 + \alpha^2 \le 0$$

implies $x_1x_2 \le 0$. Our previous argument shows $x_j \ne 0$, and the assertion follows. □

Using the previous three lemmas, it is straightforward to obtain the following result for the BVP (9.7).

Theorem 9.1 Consider (9.7) with

$$a \in \mathbb{R}, \quad \operatorname{Re}\sigma \ge 0, \quad \sigma \ne 0, \quad F \in C \cap L_2.$$

The problem has a unique solution $u \in C^2$ with $u, u_x, u_{xx} \in L_2$, and there is a constant $C_{a,\sigma}$, independent of F, with

$$\|u\|_{H^2} \le C_{a,\sigma}\|F\|. \tag{9.11}$$

Proof. Using the variable $v = u_x + au$, we can write (9.7) as a first-order system,

$$\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} -a & 1 \\ \sigma & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ F \end{pmatrix}. \tag{9.12}$$

The eigenvalues $\lambda_{1,2}$ of the system matrix, which we call A, are the roots discussed in Lemma 9.3, and there is a transformation Q with

$$Q^{-1}AQ = \Lambda = \operatorname{diag}(\lambda_1, \lambda_2).$$

In fact,

$$Q = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \sigma & \sigma \end{pmatrix}.$$

In the variables $V = (V_1, V_2)^T$ determined by

$$\begin{pmatrix} u \\ v \end{pmatrix} = Q\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

the system (9.12) becomes diagonal,

$$V_x = \Lambda V + H(x), \qquad H = Q^{-1}\begin{pmatrix} 0 \\ F \end{pmatrix}.$$

We can now use Lemmas 9.1 and 9.2 to construct and estimate an L_2-solution. Note that the boundary condition $u(0) = 0$ transforms to

$$\lambda_1 V_1(0) + \lambda_2 V_2(0) = 0. \tag{9.13}$$

By Lemma 9.2 the unique L_2-solution of $V_{2x} = \lambda_2 V_2 + H_2$ satisfies

$$\|V_2\|^2 + |V_2(0)|^2 \le C_1\|H_2\|^2, \quad C_1 = C_1(\lambda_2).$$

Using Lemma 9.1 and (9.13) to estimate V_1, we obtain

$$\|V\| \le C_2\|H\|, \quad C_2 = C_2(\lambda_{1,2}),$$

or, in the original (u, v)-variables,

$$\|u\|^2 + \|v\|^2 \le C_3\|F\|^2, \quad C_3 = C_3(\lambda_{1,2}).$$

Since $v = u_x + au$ and since we can use the differential equation to estimate u_{xx}, the inequality (9.11) follows. Clearly, by subtraction, (9.11) also yields uniqueness. □

For later reference, let us note that the Sobolev inequality (2.30) implies

$$\sup_{x \ge x_0}\left(|u(x)|^2 + |u_x(x)|^2\right) \le 2\int_{x_0}^{\infty}\left(|u(x)|^2 + |u_x(x)|^2 + |u_{xx}(x)|^2\right) dx,$$

and, by (9.11), the right-hand side tends to zero for $x_0 \to \infty$. Therefore, the solution $u(x)$ constructed in Theorem 9.1 satisfies

$$\lim_{x\to\infty} (|u(x)| + |u_x(x)|) = 0. \tag{9.14}$$

9.2. Resolvent estimate for $b < 0$

In this section we assume that the coefficient b in the parabolic equation (9.1) is negative. Recall the family of BVPs (9.4), (9.6) derived by Fourier–Laplace transformation, where the ODE (9.4) depends on the parameter

$$\sigma = s + |\omega|^2 - \mathrm{i}\rho - b, \quad \rho = \sum_{j=2}^{d} \omega_j a_j.$$

If $s = \eta + \mathrm{i}\xi, \eta \geq 0$, then the parameter σ satisfies

$$|\sigma|^2 = \left(\eta + |\omega|^2 - b\right)^2 + (\xi - \rho)^2 \geq \left(|\omega|^2 + |b|\right)^2 \geq b^2 > 0 \tag{9.15}$$

and

$$\operatorname{Re}\sigma \geq -b > 0. \tag{9.16}$$

In particular, Theorem 9.1 applies to each BVP (9.4), (9.6), and one finds

$$\|\tilde{u}(\cdot,\omega,s)\|_{H^2} \leq C_{a,\sigma}\|\tilde{G}(\cdot,\omega,s)\| \quad \text{for all } \omega \in \mathbb{R}^{d-1}, \ \operatorname{Re}s \geq 0. \tag{9.17}$$

We now use the BVP (9.4), (9.6) – or, in other notation, the BVP (9.7) – directly to sharpen the estimate and to make the dependency on $|\sigma|$ explicit. This will then allow us to apply Parseval's relation.

Theorem 9.2 Consider the BVP (9.7) with

$$a \in \mathbb{R}, \quad \operatorname{Re}\sigma \geq 0, \quad \sigma \neq 0, \quad F \in C \cap L_2,$$

and let u denote the H^2-solution constructed in Theorem 9.1. There is a constant K, independent of a, σ, and F with

$$\|u_{xx}\|^2 + |\sigma|\|u_x\|^2 + |\sigma|^2\|u\|^2 \leq K\left(1 + \frac{a^2}{|\sigma|}\right)^2 \|F\|^2. \tag{9.18}$$

Proof. Multiply (9.7) by $\bar{u}(x)$ and integrate over $0 \leq x < \infty$ to obtain

$$(u, u_{xx}) - a(u, u_x) - \sigma\|u\|^2 = (u, F).$$

Integration by parts yields

$$-\|u_x\|^2 - a(u, u_x) - \sigma\|u\|^2 = (u, F). \tag{9.19}$$

(Note that the boundary terms are zero because of (9.14) and $u(0) = 0$.) Taking the absolute value of the real part of (9.19) one finds that

$$\|u_x\|^2 + \operatorname{Re}\sigma\|u\|^2 \leq \|u\|\|F\| \tag{9.20}$$

and, therefore,

$$\|u_x\|^2 \leq \|u\|\|F\|. \tag{9.21}$$

Case 1: $\operatorname{Re}\sigma \geq |\operatorname{Im}\sigma|$, thus $|\sigma| \leq \sqrt{2}\operatorname{Re}\sigma$.
In this case, (9.20) yields

$$|\sigma|\|u\|^2 \leq \sqrt{2}\|u\|\|F\|,$$

and thus

$$|\sigma|\|u\| \leq \sqrt{2}\|F\|. \tag{9.22}$$

Also, from (9.7),

$$\|u_{xx}\| \leq |a|\|u_x\| + |\sigma|\|u\| + \|F\|. \tag{9.23}$$

Combining this with estimates (9.21) and (9.22), one obtains

$$\|u_{xx}\|^2 + |\sigma|\|u_x\|^2 + |\sigma|^2\|u\|^2 \leq K\left(1 + \frac{a^2}{|\sigma|}\right)\|F\|^2.$$

Case 2: $0 \leq \operatorname{Re}\sigma \leq |\operatorname{Im}\sigma|$, thus $|\sigma| \leq \sqrt{2}|\operatorname{Im}\sigma|$.
Taking the absolute value of the imaginary part of (9.19), one finds that

$$\begin{aligned} \frac{1}{\sqrt{2}}|\sigma|\|u\|^2 &\leq \|u\|\|F\| + |a|\|u\|\|u_x\| \\ &\leq \|u\|\|F\| + |a|\|u\|^{3/2}\|F\|^{1/2}. \end{aligned} \tag{9.24}$$

In the last estimate (9.21) has been used. From (9.24) we obtain

$$\begin{aligned} \frac{1}{\sqrt{2}}|\sigma|\|u\| &\leq \|F\| + |a|\|u\|^{1/2}\|F\|^{1/2} \\ &\leq \|F\| + \frac{1}{4}|\sigma|\|u\| + \frac{a^2}{|\sigma|}\|F\| \end{aligned}$$

and, therefore,

$$|\sigma|\|u\| \leq 2\sqrt{2}\left(1 + \frac{a^2}{|\sigma|}\right)\|F\|.$$

Combining this with (9.21) and (9.23), the desired bound (9.18) follows. □

The derivation of a strong resolvent estimate is now straightforward. We apply Theorem 9.2 to each BVP (9.4), (9.6) and recall

$$|\sigma| \geq |\omega|^2 + |b| \geq |b| > 0.$$

Therefore,

$$\begin{aligned} &K_1\|\tilde{G}(\cdot,\omega,s)\|^2 \\ &\quad\geq \|\tilde{u}_{x_1x_1}(\cdot,\omega,s)\|^2 + (|\omega|^2+1)\|\tilde{u}_{x_1}(\cdot,\omega,s)\|^2 + (|\omega|^2+1)^2\|\tilde{u}(\cdot,\omega,s)\|^2. \end{aligned}$$

By Parseval's relation we obtain the following result. (Note that the factors $|\omega|^2 + 1$ provide estimates for derivatives in the tangential directions.)

Theorem 9.3 Consider the half-space problem (9.1), (9.2), (9.3) with $b < 0$. There is a constant C, which is independent of G, such that

$$\int_0^\infty \|u(\cdot, t)\|^2_{H^2(\mathcal{H}^d)}\,\mathrm{d}t \le C \int_0^\infty \|G(\cdot, t)\|^2_{L_2(\mathcal{H}^d)}\,\mathrm{d}t \tag{9.25}$$

for all $G \in L_2(\mathcal{H}^d \times [0, \infty))$.

Remark 9.1 In our derivation of (9.25) we have used more regularity for G than $G \in L_2$. However, by applying a simple approximation argument, it is clear that the assumption $G \in L_2(\mathcal{H}^d \times [0, \infty))$ suffices.

With the same arguments as in Section 6, one can extend the basic estimate (9.25) if G is sufficiently regular and sufficiently high compatibility conditions are satisfied at $(x, t) = (0, 0)$. For example, we have

$$u_{tt} = Pu_t + G_t$$

and, if $G(x, 0) \equiv 0$, then $u_t = 0$ at $t = 0$. Therefore, (9.25) yields an estimate of u_t and its second space derivatives. Using the differential equation

$$\Delta u_t = \Delta\Delta u + \dots,$$

one obtains estimates for the fourth space derivatives of u, *etc.* As explained in Section 6, one obtains nonlinear stability for a general perturbation term $\varepsilon f(x, t, u, Du, D^2 u)$ added to (9.1).

10. Half-space problems with weak resolvent estimate

In this section we consider the parabolic initial-boundary value problem (9.1), (9.2), (9.3) with $b = 0$. Let us recall the family of ordinary BVPs on the half-line $0 \le x_1 < \infty$, derived by Fourier–Laplace transformation:

$$D_1^2 \tilde{u} + a_1 D_1 \tilde{u} - \sigma \tilde{u} = -\tilde{G}(x_1, \omega, s), \quad \tilde{u}(0, \omega, s) = 0. \tag{10.1}$$

The differential equation depends on the parameter σ,

$$\sigma = s + |\omega|^2 - \mathrm{i}\rho, \quad \rho = \sum_{j=2}^{d} \omega_j a_j, \quad \omega \in \mathbb{R}^{d-1}, \quad \mathrm{Re}\, s \ge 0. \tag{10.2}$$

If one tries to extend the results of the previous section, where we had assumed $b < 0$, to the case $b = 0$, one faces the difficulty that the crucial estimate of Theorem 9.2 becomes useless for $|\sigma| \approx 0$. In the previous section we had $|\sigma| \ge |b| > 0$, but if $b = 0$ then σ becomes zero for $s = 0, \omega = 0$.

This problem is not just technical. In fact, to obtain linear stability, one is forced to make more restrictive assumptions on the inhomogeneous term

$G(x,t)$ than smoothness and $G \in L_2(\mathcal{H}^d \times (0,\infty))$. As in Section 8, we will assume

$$G(x,t) = \sum_{j=1}^{d} D_j F_j(x,t), \quad x \in \mathcal{H}^d, \quad t \geq 0, \tag{10.3}$$

where the F_j are C^∞ functions and $F_j \in L_1(\mathcal{H}^d \times (0,\infty))$.

Note that the Fourier–Laplace transform of (10.3) is

$$\tilde{G}(x_1,\omega,s) = D_1 \tilde{F}_1(x_1,\omega,s) + \mathrm{i} \sum_{j=2}^{d} \omega_j \tilde{F}_j(x_1,\omega,s). \tag{10.4}$$

Several technical estimates are given next.

10.1. Further auxiliary results on BVPs on the half-line

In this section we use the notation

$$\|u\|^2 = \int_0^\infty |u(x)|^2 \,\mathrm{d}x, \qquad \|u\|_1 = \int_0^\infty |u(x)| \,\mathrm{d}x$$

for the L_2-norm and the L_1-norm. We start with estimates for solutions of the first-order equation $u_x = \lambda u + F(x)$.

Lemma 10.1 Let $\mathrm{Re}\,\lambda < 0$ and consider

$$u_x = \lambda u + F(x), \quad 0 \leq x < \infty, \tag{10.5}$$

where $F \in C \cap L_1$. The general solution (9.8) satisfies the estimate

$$\|u\|^2 \leq \frac{2}{|\mathrm{Re}\,\lambda|}\Big(|u(0)|^2 + \|F\|_1^2\Big).$$

Proof. For the integral term $u_2(x)$ in the general solution we have

$$|u_2(x)| \leq \int_0^x \mathrm{e}^{\mathrm{Re}\,\lambda(x-\xi)} |F(\xi)| \,\mathrm{d}\xi \leq \|F\|_1$$

and, therefore,

$$\begin{aligned}
\|u_2\|^2 &= \int_0^\infty |u_2(x)|^2 \,\mathrm{d}x \\
&\leq \|F\|_1 \int_0^\infty \int_\xi^\infty \mathrm{e}^{\mathrm{Re}\,\lambda(x-\xi)} |F(\xi)| \,\mathrm{d}x \,\mathrm{d}\xi \\
&\leq \frac{1}{|\mathrm{Re}\,\lambda|} \|F\|_1^2.
\end{aligned}$$

□

Lemma 10.2 Consider (10.5) with $\operatorname{Re}\lambda > 0, F \in C \cap L_1$. The unique L_2-solution (9.9) satisfies

$$|u(0)| \leq \|F\|_1, \qquad \|u\|^2 \leq \frac{1}{\operatorname{Re}\lambda}\|F\|_1^2.$$

Proof. The bound for $|u(0)|$ is obvious, and the bound for $\|u\|^2$ follows in the same way as in the proof of the previous lemma. □

We will also need elementary estimates for the roots $\lambda_{1,2}$ of the characteristic equation

$$\lambda^2 + a\lambda - \sigma = 0 \tag{10.6}$$

for small $|\sigma|$.

Lemma 10.3 Consider (10.6) for

$$a \in \mathbb{R}, \quad a \neq 0, \quad \operatorname{Re}\sigma \geq 0, \quad |\sigma| \leq \delta.$$

The roots are

$$\lambda_1 = -a + \mathcal{O}(|\sigma|), \qquad \lambda_2 = \frac{\sigma}{a} - \frac{\sigma^2}{a^3} + \mathcal{O}(|\sigma|^3).$$

If $\delta = \delta(a) > 0$ is small enough, then

$$|\operatorname{Re}\lambda_2| \geq \frac{1}{2}\left(\frac{\operatorname{Re}\sigma}{|a|} + \frac{|\operatorname{Im}\sigma|^2}{|a|^3}\right). \tag{10.7}$$

Proof. We only show (10.7). Setting $x = \operatorname{Re}\sigma, y = \operatorname{Im}\sigma$, the formula for λ_2 yields

$$\operatorname{Re}\lambda_2 = \frac{x}{a} - \frac{x^2 - y^2}{a^3} + \mathcal{O}(|x|^3 + |y|^3).$$

Here x/a and y^2/a^3 are not of opposite sign, and the claim follows. □

As motivated by (10.1) and (10.4), we now consider BVPs

$$u_{xx} + au_x - \sigma u = \Phi(x), \quad 0 \leq x < \infty; \quad u(0) = 0,$$

under various assumptions on $\Phi(x)$. Theorem 9.1 will guarantee existence of a unique solution $u \in H^2(0, \infty)$. The estimates in the following lemmas hold for this particular solution.

Lemma 10.4 Consider the BVP

$$u_{xx} + au_x - \sigma u = F_x(x), \quad 0 \leq x < \infty; \quad u(0) = 0,$$

where

$$a \in \mathbb{R}, \quad \operatorname{Re}\sigma \geq 0, \quad \sigma \neq 0,$$

and $F \in C^1, F, F_x \in L_2$. Then the H^2-solution satisfies

$$\|u_x\|^2 + |\sigma|\|u\|^2 \le K\left(1 + \frac{a^2}{|\sigma|}\right)\|F\|^2 \tag{10.8}$$

with a constant K independent of a, σ, and F.

Proof. Integration by parts yields

$$-\|u_x\|^2 - a(u, u_x) - \sigma\|u\|^2 = -(u_x, F). \tag{10.9}$$

Taking the absolute value of the real part, one finds that

$$\|u_x\|^2 + \operatorname{Re}\sigma\|u\|^2 \le \|u_x\|\|F\| \tag{10.10}$$

and, consequently,

$$\|u_x\| \le \|F\|. \tag{10.11}$$

Case 1: $\operatorname{Re}\sigma \ge |\operatorname{Im}\sigma|$, thus $|\sigma| \le \sqrt{2}\operatorname{Re}\sigma$.
In this case, (10.10) and (10.11) yield

$$|\sigma|\|u\|^2 \le \sqrt{2}\|u_x\|\|F\| \le \sqrt{2}\|F\|^2.$$

Together with (10.11) the estimate (10.8) follows.

Case 2: $0 \le \operatorname{Re}\sigma \le |\operatorname{Im}\sigma|$, thus $|\sigma| \le \sqrt{2}|\operatorname{Im}\sigma|$.
Taking the absolute value of the imaginary part of (10.9) and using (10.11), we find that

$$\begin{aligned} \frac{1}{\sqrt{2}}|\sigma|\|u\|^2 &\le \|u_x\|\|F\| + |a|\|u\|\|u_x\| \\ &\le \|F\|^2 + \frac{1}{4}|\sigma|\|u\|^2 + \frac{a^2}{|\sigma|}\|F\|^2. \end{aligned}$$

Again, together with (10.11) the bound (10.8) follows. □

Lemma 10.5 Consider the BVP

$$u_{xx} + au_x - \sigma u = F(x), \quad 0 \le x < \infty; \quad u(0) = 0,$$

where

$$a \in \mathbb{R}, \quad a \ne 0, \quad \operatorname{Re}\sigma \ge 0, \quad \sigma \ne 0,$$

and $F \in C \cap L_1$. There are positive constants δ and K, which depend on a, but are independent of σ and F, such that

$$\|u_x\|^2 + \|u\|^2 \le K\left(1 + \frac{1}{\tau}\right)\|F\|_1^2, \quad \tau = \frac{\operatorname{Re}\sigma}{|a|} + \frac{|\operatorname{Im}\sigma|^2}{|a|^3}, \tag{10.12}$$

if $|\sigma| \le \delta$.

Proof. Let $v = u_x + au$; thus

$$\begin{pmatrix} u \\ v \end{pmatrix}_x = A \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ F \end{pmatrix}, \quad A = \begin{pmatrix} -a & 1 \\ \sigma & 0 \end{pmatrix}.$$

The eigenvalues $\{\lambda_1, \lambda_2\}$ of A have been discussed in Lemma 10.3, namely,

$$\lambda_1 = -a + \mathcal{O}(|\sigma|), \qquad \lambda_2 = \frac{\sigma}{a} - \frac{\sigma^2}{a^3} + \mathcal{O}(|\sigma|^3). \tag{10.13}$$

Also, by Lemma 9.3 we have either $\operatorname{Re}\lambda_1 < 0 < \operatorname{Re}\lambda_2$ or $\operatorname{Re}\lambda_1 > 0 > \operatorname{Re}\lambda_2$. Our notation is set by (10.13).

We transform A to diagonal form,

$$Q^{-1}AQ = \operatorname{diag}(\lambda_1, \lambda_2),$$

using the transformation

$$\begin{aligned} Q &= \begin{pmatrix} \lambda_1 & a\lambda_2/\sigma \\ \sigma & a \end{pmatrix} = \begin{pmatrix} -a & 1 \\ 0 & a \end{pmatrix} + \mathcal{O}(|\sigma|), \\ Q^{-1} &= a^{-2} \begin{pmatrix} -a & 1 \\ 0 & a \end{pmatrix} + \mathcal{O}(|\sigma|). \end{aligned}$$

For the variables $V = (V_1, V_2)^T$ defined by

$$\begin{pmatrix} u \\ v \end{pmatrix} = QV \tag{10.14}$$

one obtains scalar equations

$$V_{1x} = \lambda_1 V_1 + H_1, \qquad V_{2x} = \lambda_2 V_2 + H_2,$$

where

$$H = Q^{-1} \begin{pmatrix} 0 \\ F \end{pmatrix}, \qquad \|H\|_1 \le C_a \|F\|_1.$$

Note that the boundary condition $u(0) = 0$ transforms to

$$\lambda_1 V_1(0) + \frac{a\lambda_2}{\sigma} V_2(0) = 0. \tag{10.15}$$

By (10.13), the coefficients are bounded away from zero and infinity.

Case 1: $a < 0$, thus $\operatorname{Re}\lambda_1 > 0 > \operatorname{Re}\lambda_2$.
By Lemma 10.2 we have

$$|V_1(0)| \le \|F\|_1, \qquad \|V_1\|^2 \le \frac{1}{\operatorname{Re}\lambda_1} \|F\|_1^2. \tag{10.16}$$

Lemma 10.1, (10.16) and (10.15) yield

$$\|V_2\|^2 \le \frac{K_a}{|\operatorname{Re}\lambda_2|} \|F\|_1^2.$$

Finally, by Lemma 10.3,

$$\frac{1}{|\operatorname{Re}\lambda_2|} \le \frac{2}{\tau}$$

and, therefore,

$$\|V\|^2 \le K_a \left(1 + \frac{1}{\tau}\right) \|F\|_1^2.$$

Because of (10.14) and $v = u_x + au$ the desired bound follows.

Case 2: $a > 0$, thus $\operatorname{Re}\lambda_1 < 0 < \operatorname{Re}\lambda_2$.
By Lemma 10.2 we have

$$|V_2(0)| \le \|F\|_1, \qquad \|V_2\|^2 \le \frac{1}{\operatorname{Re}\lambda_2} \|F\|_1^2. \tag{10.17}$$

Lemma 10.1, (10.17) and (10.15) yield

$$\|V_1\|^2 \le K_a \|F\|_1^2$$

since $|\operatorname{Re}\lambda_1| \approx |a|$. The remaining arguments are the same as in Case 1. □

Lemma 10.6 Let $a > 0$ and let $\delta = \delta(a) > 0$ be determined as in Lemma 10.5. Consider the BVP

$$u_{xx} + au_x - \sigma u = F_x(x), \quad 0 \le x < \infty; \quad u(0) = 0,$$

under the assumptions

$$\operatorname{Re}\sigma \ge 0, \quad 0 < |\sigma| \le \delta, \quad F \in C^1 \cap L_1.$$

There is a constant $K = K_a$, independent of σ and F, such that

$$\|u_x\|^2 + \|u\|^2 \le K\left(\|F\|^2 + \|F\|_1^2\right).$$

Proof. Define $v(x)$ to be the solution of

$$v_x + av = F, \quad v(0) = 0.$$

Using Lemma 9.1 we find that

$$\|v_x\|^2 + \|v\|^2 \le K\|F\|^2,$$

and the estimate $\|v\|_1 \le \frac{1}{a}\|F\|_1$ follows easily from

$$v(x) = \int_0^x e^{-a(x-\xi)} F(\xi)\, d\xi.$$

The difference $w = u - v$ satisfies

$$w_{xx} + aw_x - \sigma w = \sigma v, \quad w(0) = 0.$$

By Lemma 10.5,

$$\|w_x\|^2 + \|w\|^2 \le K\left(1 + \frac{1}{\tau}\right) |\sigma|^2 \|v\|_1^2,$$

where

$$\tau = \frac{\operatorname{Re}\sigma}{a} + \frac{|\operatorname{Im}\sigma|^2}{a}.$$

If $\sigma = x + iy \quad (x, y \in \mathbb{R})$, then

$$|\sigma|^2 = x^2 + y^2 \leq \delta x + y^2 \leq C\tau.$$

Thus we find

$$\|w_x\|^2 + \|w\|^2 \leq K\|v\|_1^2,$$

and, together with the estimates for v, the lemma is proved. □

Lemma 10.7 Let $a < 0$ and let $\delta = \delta(a) > 0$ be determined as in Lemma 10.5. Consider the BVP

$$u_{xx} + au_x - \sigma u = F_x(x), \quad 0 \leq x < \infty; \quad u(0) = 0,$$

under the assumptions

$$\operatorname{Re}\sigma \geq 0, \quad 0 < |\sigma| \leq \delta, \quad F \in C^1 \cap L_1.$$

There is a constant $K = K_a$, independent of σ and F, such that

$$\|u_x\|^2 + \|u\|^2 \leq K\|F\|^2 + \frac{K}{|\operatorname{Re}\lambda_2|}\|F\|_1^2.$$

Here $\lambda_2 = \frac{\sigma}{a} + \mathcal{O}(|\sigma|^2)$ is small.

Proof. Define $v(x)$ to be the L_2-solution of

$$v_x + av = F, \quad 0 \leq x < \infty;$$

thus

$$v(x) = -\int_x^\infty \mathrm{e}^{-a(x-\xi)} F(\xi)\,\mathrm{d}\xi. \tag{10.18}$$

Using Lemma 9.2 we find that

$$\|v_x\|^2 + \|v\|^2 \leq K\|F\|^2,$$

and the estimates

$$|v(0)| \leq \|F\|_1, \quad \|v\|_1 \leq \frac{1}{|a|}\|F\|_1$$

follow easily from (10.18).

The difference $w = u - v$ satisfies

$$w_{xx} + aw_x - \sigma w = \sigma v, \quad w(0) = -v(0).$$

We write $w = w_1 + w_2$ where w_1 and w_2 solve

$$\begin{aligned} w_{1xx} + aw_{1x} - \sigma w_1 &= \sigma v, \quad w_1(0) = 0, \\ w_{2xx} + aw_{2x} - \sigma w_2 &= 0, \quad w_2(0) = -v(0). \end{aligned}$$

The estimate of w_1 proceeds in the same way as the estimate of w in the proof of the previous lemma; thus

$$\|w_{1x}\|^2 + \|w_1\|^2 \le K\|F\|_1^2.$$

Finally, we have $w_2(x) = -v(0)\mathrm{e}^{\lambda_2 x}$ and, therefore,

$$\|w_2\|^2 = \frac{|v(0)|^2}{2|\mathrm{Re}\,\lambda_2|} \le \frac{\|F\|_1^2}{2|\mathrm{Re}\,\lambda_2|}.$$

From these estimates the lemma follows. □

10.2. Weak resolvent estimate

Consider the initial-boundary value problem

$$u_t = \Delta u + \sum_{j=1}^{d} a_j D_j u + \sum_{j=1}^{d} D_j F_j(x,t), \quad x \in \mathcal{H}^d, \quad t \ge 0, \tag{10.19}$$

$$u(x,0) = 0, \quad x \in \mathcal{H}^d; \qquad u(x,t) = 0, \quad x \in \partial\mathcal{H}^d, \ t \ge 0, \tag{10.20}$$

with real constants a_j and assume $a_1 \ne 0$.

For the one-dimensional hyperbolic problem

$$v_t = a_1 D_1 v + F(x_1, t), \quad 0 \le x_1 < \infty, \quad t \ge 0,$$

the cases $a_1 > 0$ and $a_1 < 0$ are quite different. If $a_1 > 0$, then v is an outgoing characteristic variable, no boundary condition can be given at $x_1 = 0$, and one can easily estimate v by F. In contrast, if $a_1 < 0$, then v is an in-going characteristic variable, one needs a boundary condition at $x_1 = 0$, and estimates of v by F are more restrictive. As we will see, the two cases $a_1 > 0$ and $a_1 < 0$ also lead to different stability conditions for (10.19), (10.20).

Fourier transformation in the tangential variables $x_2, \dots, x_d$ and Laplace transformation in time leads to the following family of ordinary BVPs for $\tilde{u} = \tilde{u}(x_1, \omega, s)$:

$$D_1^2 \tilde{u} + a_1 D_1 \tilde{u} - \sigma \tilde{u} = -D_1 \tilde{F}_1 - \mathrm{i} \sum_{j=2}^{d} \omega_j \tilde{F}_j =: \Phi(x_1, \omega, s), \quad 0 \le x_1 < \infty, \tag{10.21}$$

with boundary condition

$$\tilde{u}(0, \omega, s) = 0. \tag{10.22}$$

Here

$$\sigma = s + |\omega|^2 - \mathrm{i} \sum_{j=2}^{d} \omega_j a_j, \quad \omega \in \mathbb{R}^{d-1}, \quad \mathrm{Re}\, s \ge 0. \tag{10.23}$$

The next lemma gives the essential estimate of $\tilde{u}$ by $\tilde{F}$ in the case $a_1 > 0$. Its proof follows by collecting the relevant auxiliary results of Sections 9.1 and 10.1.

Lemma 10.8 Let $a_1 > 0, \operatorname{Re} s \geq 0, \omega \in \mathbb{R}^{d-1}$, and let $\delta = \delta(a_1) > 0$ be determined as in Lemma 10.5. If $|\sigma| \geq \delta$, then

$$\begin{aligned} &\int_0^\infty \Big(|D_1 \tilde{u}(x_1, \omega, s)|^2 + (|\omega|^2 + 1)|\tilde{u}(x_1, \omega, s)|^2 \Big)\, \mathrm{d}x_1 \\ &\qquad \leq \; K \int_0^\infty |\tilde{F}(x_1, \omega, s)|^2\, \mathrm{d}x_1 \\ &\qquad = \; K \|\tilde{F}(\cdot, \omega, s)\|^2. \end{aligned} \tag{10.24}$$

If $0 < |\sigma| \leq \delta$, then

$$\begin{aligned} &\int_0^\infty \Big(|D_1 \tilde{u}(x_1, \omega, s)|^2 + (|\omega|^2 + 1)|\tilde{u}(x_1, \omega, s)|^2 \Big)\, \mathrm{d}x_1 \\ &\qquad \leq \; K \|\tilde{F}(\cdot, \omega, s)\|^2 + K \left(\int_0^\infty |\tilde{F}(x_1, \omega, s)|\, \mathrm{d}x_1 \right)^2. \end{aligned} \tag{10.25}$$

Proof. Write the right-hand side of (10.21) as $\Phi = \Phi_1 + \Phi_2$,

$$\Phi_1 = -D_1 \tilde{F}_1, \qquad \Phi_2 = -\mathrm{i} \sum_{j=2}^d \omega_j \tilde{F}_j,$$

and decompose $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$ accordingly. We treat four cases separately.

Case 1: $|\sigma| \geq \delta$; estimate of $\tilde{u}_1$.
Note that

$$|\omega|^2 + 1 \leq |\sigma| + 1 \leq \left(1 + \frac{1}{\delta}\right) |\sigma|.$$

By Lemma 10.4,

$$\|D_1 \tilde{u}_1\|^2 + (|\omega|^2 + 1)\|\tilde{u}_1\|^2 \leq K \|\tilde{F}_1\|^2.$$

Case 2: $|\sigma| \geq \delta$; estimate of $\tilde{u}_2$.
By Theorem 9.2,

$$\|D_1 \tilde{u}_2\|^2 + \left(|\omega|^2 + 1\right) \|\tilde{u}_2\|^2 \leq \frac{K}{|\sigma|} \|\Phi_2\|^2 \leq K \sum_{j=2}^d \|\tilde{F}_j\|^2. \tag{10.26}$$

Here we have used that $|\sigma| \geq |\omega|^2$.

Case 3: $0 < |\sigma| \leq \delta$; estimate of $\tilde{u}_1$.
First note that $|\omega|^2 \leq |\sigma| \leq \delta$. Therefore, by Lemma 10.6,

$$\|D_1 \tilde{u}_1\|^2 + \left(|\omega|^2 + 1\right) \|\tilde{u}_1\|^2 \leq K \Big(\|\tilde{F}_1\|^2 + \|\tilde{F}_1\|_1^2 \Big). \tag{10.27}$$

Case 4: $0 < |\sigma| \le \delta$; estimate of $\tilde{u}_2$.
By Lemma 10.5,

$$\|D_1\tilde{u}_2\|^2 + \left(|\omega|^2 + 1\right) \|\tilde{u}_2\|^2 \le K \left(1 + \frac{1}{\tau}\right) \|\Phi_2\|^2 \tag{10.28}$$

with

$$\tau \ge \frac{\operatorname{Re}\sigma}{a_1} \ge \frac{|\omega|^2}{a_1}.$$

As before, $\|\Phi_2\|^2 \le |\omega|^2\|\tilde{F}\|^2$. Therefore, the right-hand side of (10.28) is bounded by $K\|\tilde{F}\|^2$.

Collecting these estimates, we have proved the lemma. □

It will be convenient to use the following cut-off function $\chi(\omega, s)$,

$$\chi(\omega, s) = \begin{cases} 1 & \text{if } |\sigma| \le \delta, \\ 0 & \text{if } |\sigma| > \delta, \end{cases}$$

where $\sigma = \sigma(\omega, s)$ is defined by (10.23). Also, from the definition of the Fourier–Laplace transform,

$$|\tilde{F}_j(x_1, \omega, s)| \le \int_0^\infty \int_{\mathbb{R}^{d-1}} |F_j(x_1, x_-, t)| \,\mathrm{d}x_- \,\mathrm{d}t,$$

and integration over $0 \le x_1 < \infty$ yields

$$\|\tilde{F}(\cdot, \omega, s)\|_1 \le C\|F\|_{L_1(\mathcal{H}^d \times (0,\infty))}.$$

Therefore, the two estimates proved in Lemma 10.8 can be summarized by the following inequality:

$$\begin{aligned} &\int_0^\infty \left(|D_1\tilde{u}(x_1, \omega, s)|^2 + (|\omega|^2 + 1)|\tilde{u}(x_1, \omega, s)|^2\right) \mathrm{d}x_1 \\ &\qquad \le \quad K\|\tilde{F}(\cdot, \omega, s)\|^2 + K\chi(\omega, s)\|F\|^2_{L_1(\mathcal{H}^d \times (0,\infty))}. \end{aligned} \tag{10.29}$$

Here $\omega \in \mathbb{R}^{d-1}$ and $s \in \mathbb{C}$ with $\operatorname{Re} s \ge 0$ are arbitrary, except that we require $(\omega, s) \ne (0, 0)$.

(If $\omega = 0, s = 0$, then $\sigma = 0$. However, for $\sigma = 0$ the solution of the BVPs treated in Lemma 10.4, *etc.*, is not in L_2, in general. In the application of Parseval's relation, the single exceptional point $\sigma = 0$ causes no problem.)

To estimate $u(x, t)$, instead of $\tilde{u}$, we apply Parseval's relation with $\eta = 0$. First, Parseval's relation yields

$$\begin{aligned} &\int_0^\infty \|u(\cdot, t)\|^2_{H^1(\mathcal{H}^d)} \,\mathrm{d}t \\ &\le \; C\int_{-\infty}^\infty \int_{\mathbb{R}^{d-1}} \int_0^\infty \left(|D_1\tilde{u}(x_1, \omega, s)|^2 + (|\omega|^2 + 1)|\tilde{u}(x_1, \omega, s)|^2\right) \mathrm{d}x_1 \,\mathrm{d}\omega \,\mathrm{d}\xi. \end{aligned} \tag{10.30}$$

Integrating (10.29) over $\omega \in \mathbb{R}^{d-1}$ and $s = \mathrm{i}\xi$, $-\infty < \xi < \infty$, we find that

$$\int_0^\infty \|u(\cdot,t)\|^2_{H^1(\mathcal{H}^d)}\,\mathrm{d}t \le K \int_0^\infty \|F(\cdot,t)\|^2\,\mathrm{d}t + KJ\|F\|^2_{L_1(\mathcal{H}^d\times(0,\infty))} \quad (10.31)$$

with

$$J = \int_{\mathbb{R}^{d-1}} \int_{-\infty}^\infty \chi(\omega, \mathrm{i}\xi)\,\mathrm{d}\omega\,\mathrm{d}\xi. \quad (10.32)$$

It is clear that J is finite since $\chi(\omega,\xi) = 0$ if $|\omega| \ge c_\delta$ or $|\xi| \ge c_\delta$.

Therefore, we have proved the following resolvent estimate.

Theorem 10.1 Consider the initial-boundary value problem (10.19) and (10.20) with $a_1 > 0$ and

$$F \in L_2(\mathcal{H}^d \times (0,\infty)) \cap L_1(\mathcal{H}^d \times (0,\infty)).$$

Then we have

$$\int_0^\infty \|u(\cdot,t)\|^2_{H^1(\mathcal{H}^d)}\,\mathrm{d}t \le C\left(\int_0^\infty \|F(\cdot,t)\|^2\,\mathrm{d}t + \left(\int_0^\infty \|F(\cdot,t)\|_1\,\mathrm{d}t\right)^2\right) \quad (10.33)$$

where C does not depend on F.

In case $a_1 < 0$ we obtain the same result if the number of space dimensions is $d \ge 3$.

Theorem 10.2 Consider the initial-boundary value problem (10.19) and (10.20) with $a_1 < 0$. Under the same assumptions on F as in Theorem 10.1 the estimate (10.33) holds if $d \ge 3$.

Proof. If $a_1 < 0$, Lemma 10.8 needs to be modified. Using the notation of the proof of Lemma 10.8, we have, by Lemma 10.7,

Case 3a: $0 < |\sigma| \le \delta$; estimate of $\tilde{u}_1$ if $a_1 < 0$:

$$\|D_1\tilde{u}_1\|^2 + (|\omega|^2 + 1)\|\tilde{u}_1\|^2 \le K\|\tilde{F}_1\|^2 + \frac{K}{|\mathrm{Re}\,\lambda_2|}\|\tilde{F}_1\|^2.$$

All other estimates in the proof of Lemma 10.8 remain unchanged. Therefore, in the estimate corresponding to (10.29) the factor $\chi(\omega, s)$ has to be replaced by

$$\frac{\chi(\omega,s)}{|\mathrm{Re}\,\lambda_2|}$$

and, consequently, instead of the integral J (see (10.32)) one has to consider

$$J' = \int_{\mathbb{R}^{d-1}} \int_{-\infty}^\infty \frac{\chi(\omega,\mathrm{i}\xi)}{|\mathrm{Re}\,\lambda_2(\omega,\mathrm{i}\xi)|}\,\mathrm{d}\omega\,\mathrm{d}\xi. \quad (10.34)$$

By Lemma 10.3,

$$|\operatorname{Re}\lambda_2| \geq \frac{1}{2}\left(\frac{|\omega|^2}{|a_1|} + \frac{(\xi-\rho)^2}{|a_1|^3}\right), \quad \rho = \sum_{j=2}^{d} \omega_j a_j.$$

Since the integral

$$\int_{|\omega|\leq 1}\int_{-\infty}^{\infty} \frac{d\xi\, d\omega}{|\omega|^2+\xi^2} \quad (\omega \in \mathbb{R}^{d-1})$$

is finite for $d-1 \geq 2$, the integral J' is also finite, and the theorem is proved. □

As in Section 6, the basic estimates of Theorem 10.1 and 10.2 can be extended. Assuming compatibility conditions are satisfied, one may assume $F_j(x,0) \equiv 0$. Then u_t satisfies

$$u_{tt} = Pu_t + \sum_j D_j F_{jt}, \quad u_t = 0 \quad \text{at} \quad t = 0,$$

and one obtains an estimate for u_t and its first space derivatives, *etc.* The perturbation terms for which one obtains nonlinear stability are described in Section 8.3.

11. Eigenvalue and spectral conditions for parabolic problems on the line

In Section 6 we have considered parabolic problems

$$u_t = Pu + \varepsilon f(x,t,u,u_x,u_{xx}) + F(x,t)$$

in a finite interval $0 \leq x \leq 1$ and have shown that the eigenvalue condition for P is necessary and sufficient for nonlinear stability. Here the eigenvalue condition for P requires that the problem

$$P\phi = \lambda\phi, \qquad R\phi = 0,$$

has no eigenvalue λ with $\operatorname{Re}\lambda \geq 0$. (With $R\phi = 0$ we denote homogeneous boundary conditions; see Section 6.) Such a result is not restricted to bounded intervals, but can be generalized to parabolic problems in bounded domains in any number of space dimensions.

On the other hand, the result cannot be extended to unbounded domains without further restrictions. For example, consider the problem on the half-line,

$$u_t = u_{xx} + au_x + G(x,t), \quad 0 \leq x < \infty,$$

with initial and boundary conditions

$$u(x,0)=0, \quad x \geq 0; \qquad u(0,t)=0, \quad t \geq 0,$$

and $a \in \mathbb{R}$. It is easy to see that the corresponding eigenvalue problem

$$\phi_{xx}+a\phi_x=\lambda\phi, \quad \phi(0)=0, \quad \phi \in L_2(0,\infty),$$

has no eigenvalue λ with $\operatorname{Re}\lambda \geq 0$. (Use Lemma 9.3.) However, by Theorem 10.1, we can only obtain a *weak* resolvent estimate, and need the extra assumptions $a>0$ and $G(x,t)=F_x(x,t)$.

We shall now discuss cases for parabolic problems on the real line $-\infty < x < \infty$ for which the eigenvalue condition for P does imply a *strong* resolvent estimate. The results can be extended to multi-dimensions.

The main principle is to reduce the problem on the infinite line to a problem on a finite interval $-L \leq x \leq L$ by expressing the 'tails' of the problem as boundary conditions at $x=\pm L$. (In numerical computations a similar process is often used when one introduces an artificial boundary for infinite domain problems; see, for example, Hagstrom and Keller (1986).)

To be more specific, consider a parabolic system

$$u_t=Pu+F(x,t), \quad x \in \mathbb{R}, \quad t \geq 0,$$

with initial condition

$$u(x,0)=0, \quad x \in \mathbb{R}.$$

Here

$$Pu=(A(x)u_x)_x+B(x)u_x+C(x)u \tag{11.1}$$

with smooth matrix functions $A(x), B(x), C(x)$ taking values in $\mathbb{C}^{n\times n}$, and parabolicity of $u_t=Pu$ requires

$$A(x)=A^*(x) \geq \alpha I > 0, \quad x \in \mathbb{R}.$$

Also, we assume that the coefficients $A(x), B(x), C(x)$ converge exponentially fast to constant matrices $A_\pm, B_\pm, C_\pm$ as $x \to \pm\infty$. For example,

$$|A(x)-A_+| \leq K\mathrm{e}^{-\gamma x}, \quad x \geq 0,$$

with $\gamma > 0$. The properties of the resulting constant coefficient equations

$$u_t=P_+u \quad \text{and} \quad u_t=P_-u$$

are very important. (We set $P_+u = A_+u_{xx}+B_+u_x+C_+u$, and P_- is defined correspondingly.) The following lemma relates properties of the symbols $\hat{P}_\pm(\mathrm{i}\omega)$, $\omega \in \mathbb{R}$, to decay and growth properties of the solutions of the homogeneous equations

$$P_\pm u - su = 0.$$

Lemma 11.1 Let $A_0, B_0, C_0 \in \mathbb{C}^{n\times m}$ denote constant matrices with $A_0 = A_0^* \geq \alpha I > 0$, and let

$$\hat{P}_0(\kappa) = \kappa^2 A_0 + \kappa B_0 + C_0, \quad \kappa \in \mathbb{C},$$

denote the symbol of

$$P_0 u = A_0 u_{xx} + B_0 u_x + C_0 u.$$

The following two conditions are equivalent.

(1) There is a constant $\delta_1 > 0$, such that

$$\det(\hat{P}_0(\mathrm{i}\omega) - \lambda I) = 0 \quad \text{and} \quad \omega \in \mathbb{R} \tag{11.2}$$

implies $\operatorname{Re}\lambda \leq -\delta_1 < 0$.

(2) There are constants $\gamma > 0$ and $\delta_2 > 0$, such that

$$\det(\hat{P}_0(\kappa) - sI) = 0 \quad \text{and} \quad \operatorname{Re} s \geq -\delta_2 \tag{11.3}$$

implies $|\operatorname{Re}\kappa| \geq \gamma > 0$.

Proof. First assume condition (1) and let $\operatorname{Re} s \geq -\delta_1/2$. If κ solves (11.3), then (1) implies $\operatorname{Re}\kappa \neq 0$. Fix any $c > 0$. If $|s| \leq c$ and $\operatorname{Re} s \geq -\delta_1/2$ then, by continuity, there exists $\gamma = \gamma(c) > 0$ with $|\operatorname{Re}\kappa| \geq \gamma > 0$ for all solutions κ of (11.3). For large s, $|s| \geq c$, the roots κ of (11.3) satisfy to leading order

$$\det(\kappa^2 A - sI) = 0.$$

If $\alpha_1, \ldots, \alpha_n$ denote the eigenvalues of A, then $\alpha_j \geq \alpha$ and

$$\kappa \approx \pm\sqrt{\frac{s}{\alpha_j}}. \tag{11.4}$$

Therefore, if $\operatorname{Re} s \geq -1$ and $|s|$ is large, then $|\operatorname{Re}\kappa| \geq 1$. This shows that (1) implies (2).

Now assume condition (2) and let ω and λ satisfy (11.2). By (2) we have $\operatorname{Re}\lambda < -\delta_2$, and the lemma is proved. □

Under the assumptions of the lemma, equation (11.3) for κ, which is sometimes called the dispersion relation, has $2n$ roots κ_j with

$$\operatorname{Re}\kappa_j \leq -\gamma, \quad j = 1, \ldots, n \quad \text{and} \quad \operatorname{Re}\kappa_j \geq \gamma, \quad j = n+1, \ldots, 2n.$$

This follows from (11.4) and continuous dependence of the roots on s. In the language of dynamical systems, the homogeneous equation $P_0 u - su = 0$ has an exponential dichotomy on $\mathbb{R}$, which is uniform for $\operatorname{Re} s \geq -\delta_2$.

Now consider the variable coefficient operator P given by (11.1) and recall $A(x) \to A_\pm$ as $x \to \pm\infty$, *etc.* We say that P satisfies the *strong spectral condition* if both operators, P_+ and P_-, satisfy the conditions of the previous lemma; that is, there exists $\delta_1 > 0$ such that, for all $\omega \in \mathbb{R}$,

$$\lambda \in \sigma(\hat{P}_+(\mathrm{i}\omega)) \cup \sigma(\hat{P}_-(\mathrm{i}\omega)) \quad \text{implies} \quad \operatorname{Re}\lambda \le -\delta_1 < 0. \tag{11.5}$$

Also, P satisfies the *weak spectral condition* if, for all $\omega \in \mathbb{R}$,

$$\lambda \in \sigma(\hat{P}_+(\mathrm{i}\omega)) \cup \sigma(\hat{P}_-(\mathrm{i}\omega)) \quad \text{implies} \quad \operatorname{Re}\lambda \le 0. \tag{11.6}$$

To give a simple example, consider the scalar operator

$$Pu = u_{xx} + au_x + bu, \quad a, b \in \mathbb{R},$$

with $\hat{P}(\mathrm{i}\omega) = -\omega^2 + \mathrm{i}a\omega + b$. We see that P satisfies the strong spectral condition if $b < 0$, but only the weak spectral condition if $b = 0$. Neither condition holds for $b > 0$.

The following result can be proved.

Theorem 11.1 Assume that P has the form (11.1) and satisfies the strong spectral condition. If the eigenvalue problem

$$P\phi = \lambda\phi, \quad \phi \in L_2(\mathbb{R}), \tag{11.7}$$

has no eigenvalue λ with $\operatorname{Re}\lambda \ge 0$, then a strong resolvent estimate (as in Section 6) holds. One obtains nonlinear stability with perturbations $\varepsilon f(x, t, u, u_x, u_{xx})$ as in Section 6.

The main part of the proof consists of reducing the infinite interval to a finite one; Lemma 11.1 is one step in showing that such a reduction is possible. For the finite-interval problem, the techniques of Section 6 apply. We refer to Kreiss, Kreiss and Petersson (1994*a*) for details.

Theorem 11.1 can be used to discuss stability of travelling; see Kreiss et al. (1994*a*) and the literature cited there. (There is a technical problem resulting from a zero eigenvalue of P, which corresponds to shifting the travelling wave. Using a projection as in Henry (1981) and Kreiss et al. (1994*a*), this problem can be overcome, however.)

In many interesting cases only the weak spectral condition (11.6) is satisfied. Then, for scalar equations, one can often use exponentially weighted norms as pioneered by Sattinger (1976). Instead of using these norms, one can also make a change of the dependent variable u so that, in the new variable, the strong spectral condition (11.5) is satisfied.

For systems of equations, however, this approach does not generally give the desired results. Nevertheless, stability has recently been shown for rather general travelling shock waves governed by systems of conservation laws (Kreiss and Kreiss 1997). Essentially, the only requirements are the weak spectral condition and the condition that (11.7) has no eigenvalue λ with $\operatorname{Re}\lambda \ge 0$ except $\lambda = 0$. The inhomogeneous term and the nonlinear perturbation must satisfy the restrictions of Section 8.

Acknowledgement
This collaboration was made possible by the visit of the second author to the Mittag-Leffler Institute in Djorsholm, Sweden, which was supported by the Swedish Academy of Science. In particular, we would like to thank Professors B. Engquist and K. O. Widman for the excellent arrangements.

REFERENCES

T. Hagstrom and H. B. Keller (1986), 'Exact boundary conditions at an artificial boundary for partial differential equations in cylinders', *SIAM J. Math. Anal.* **17**, 322–341.

T. Hagstrom and J. Lorenz (1995), 'All-time existence of smooth solutions to PDEs of mixed type and the invariant subspace of uniform states', *Adv. Appl. Math.* **16**, 219–257.

D. Henry (1981), *Geometric Theory of Semilinear Parabolic Equations*, Springer, Berlin.

S. Kawashima (1987), 'Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications', *Proceedings of the Royal Society of Edinburgh* **106A**, 169–194.

S. Klainerman and G. Ponce (1983), 'Global, small amplitude solutions to nonlinear evolution equations', *Comm. Pure Appl. Math.* **36**, 133–141.

G. Kreiss and H.-O. Kreiss (1997), Stability of systems of viscous conservation laws, Technical Report 97-54, UCLA CAM Report.

G. Kreiss, H.-O. Kreiss and J. Lorenz (1998*a*), 'On stability of conservation laws', *SIAM J. Math. Anal.* To appear.

G. Kreiss, H.-O. Kreiss and J. Lorenz (1998*b*), 'On the use of symmetrizers for stability analysis'. In preparation.

G. Kreiss, H.-O. Kreiss and A. Petersson (1994*a*), 'On the convergence to steady state of solutions of nonlinear hyperbolic-parabolic systems', *SIAM J. Numer. Anal.* **31**, 1577–1604.

G. Kreiss, A. Lundbladh and D. Henningson (1994*b*), 'Bounds for threshold amplitudes in subcritical shear flow', *J. Fluid Mech.* **270**, 175–198.

H.-O. Kreiss and J. Lorenz (1989), *Initial-Boundary Value Problems and the Navier–Stokes Equations*, Academic, New York.

H.-O. Kreiss and S. H. Lui (1997), Nonlinear stability of time dependent differential equations, Technical Report 97–43, UCLA CAM Report.

H.-O. Kreiss, O. Ortiz and O. Reula (1998*c*), 'Stability of quasi-linear hyperbolic dissipative systems', *J. Diff. Eqns.* To appear.

A. M. Lyapunov (1956), *General Problem of the Stability of Motion, Collected Works*, Vol. 2, Izdat. Akad. Nauk SSSR, Moscow. (Russian).

A. Matsumura and T. Nishida (1979), 'The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids', *Proc. Jap. Acad., Ser. A* **55**, *337–341.*

R. Racke (1992), *Lectures on Nonlinear Evolution Equations: Initial Value Problems*, Vieweg Verlag, Braunschweig, Wiesbaden.

V. A. Romanov (1973), 'Stability and plane-parallel Couette flow', *Funktsional'nyi Analizi Ego Prilozhaniya* **7**, 62–73. Translated in *Funct. Anal. Appl.* **7**, 137–146.

D. H. Sattinger (1976), 'On the stability of waves of nonlinear parabolic systems', *Adv. Math.* **22**, 312–355.

J. Shatah (1982), 'Global existence of small solutions to nonlinear evolution equations', *J. Diff. Eqns* **46**, 409–425.

W. Strauss (1981), 'Nonlinear scattering theory at low energy', *J. Funct. Anal.* **41**, 110–133.

L. N. Trefethen (1997), 'Pseudospectra of linear operators', *SIAM Rev.* **39**, 383–406.